Blasius Solution for a Flat Plate Boundary Layer

The first exact solution to the laminar boundary layer equations, discovered by Blasius (1908), was for a simple constant value of $U(s)$ and pertains to the case of a uniform stream of velocity, U, encountering an infinitely thin flat plate set parallel with that stream as shown in Figure 1:

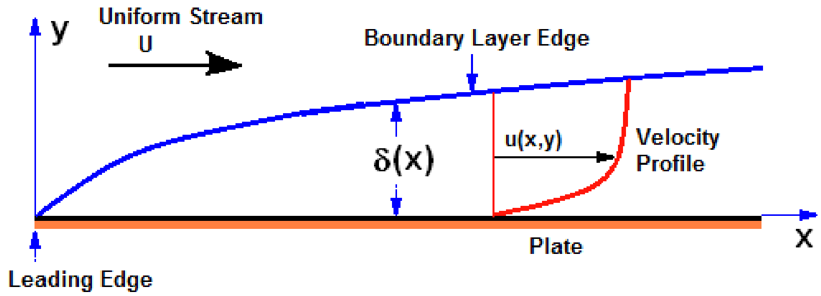

Figure 1: Blasius flow of a boundary layer on a flat plate

Thus the laminar boundary layer equations (Bjb20) and (Bjb21) become

$$\frac{\partial u}{\partial s} + \frac{\partial v}{\partial n} = 0 \tag{Bjd1}$$

$$u\frac{\partial u}{\partial s} + v\frac{\partial u}{\partial n} = \nu\frac{\partial^2 u}{\partial n^2} \tag{Bjd2}$$

and, using the stream function, ψ, given by

$$u = \frac{\partial \psi}{\partial n} \qquad \text{and} \qquad v = -\frac{\partial \psi}{\partial s} \tag{Bjd3}$$

the single governing equation that Blasius set out to solve is

$$\frac{\partial \psi}{\partial n}\frac{\partial^2 \psi}{\partial s \partial n} - \frac{\partial \psi}{\partial s}\frac{\partial^2 \psi}{\partial n^2} = \nu\frac{\partial^3 \psi}{\partial n^3} \tag{Bjd4}$$

with the following boundary conditions:

1. $\psi = 0$ on the solid surface, $n = 0$, since v=0
2. $\partial\psi/\partial n = 0$ on the solid surface, $n = 0$ since u=0
3. $\partial\psi/\partial n \to U$ as $n \to \infty$ since $u \to U$

Notice that we have avoided using δ in the formulation of the mathematical problem. The definition and evaluation of the boundary layer thickness will follow from the results.

The Blasius solution is best presented as an example of a similarity solution to the non-linear, partial differential equation (Bjd4). In a similarity solution we seek a similarity variable (here symbolized by η) which is a function of s and n such that the unknown ψ may be written as a function of the single variable η. If this is possible then the problem is reduced to finding the solution to an *ordinary* differential equation for $\psi(\eta)$. Of course, it may not be possible to find such a similarity variable in which case no such solution exists. Consequently the key step is in finding a possible η.

Let us speculate that a successful similarity variable for the present problem takes the form

$$\eta = An/s^k \tag{Bjd5}$$

where A and k are unknown constants. Then

$$\frac{\partial \eta}{\partial n} = \frac{A}{s^k} \qquad \text{and} \qquad \frac{\partial \eta}{\partial s} = -\frac{Akn}{s^{k+1}} = -\frac{k\eta}{s} \tag{Bjd6}$$

We further speculate that the solution, $\psi(\eta)$, to equation (Bjd4) takes the form

$$\psi = Bs^k F(\eta) \tag{Bjd7}$$

where B is another unknown constant and $F(\eta)$ is an unknown function. It follows that

$$u = \frac{\partial \psi}{\partial n} = Bs^k \frac{dF}{d\eta}\frac{\partial \eta}{\partial n} = AB\,\frac{dF}{d\eta} \tag{Bjd8}$$

and it is therefore convenient and without loss of generality to choose $AB = U$ so that

$$u = U\frac{dF}{d\eta} \tag{Bjd9}$$

Note that this choice is merely for convenience and could have been omitted from the solution. Next we observe that the following derivatives that appear in the governing equation (Bjd4) take these forms:

$$v = \frac{\partial \psi}{\partial s} = -\frac{kUs^{k-1}}{A}\left\{\eta\frac{dF}{d\eta} - F\right\} \tag{Bjd10}$$

$$\frac{\partial^2 \psi}{\partial n^2} = \frac{\partial u}{\partial n} = \frac{UA}{s^k}\frac{d^2F}{d\eta^2} \quad ; \quad \frac{\partial^2 \psi}{\partial n \partial s} = \frac{\partial u}{\partial s} = -\frac{kUAn}{s^{k+1}}\frac{d^2F}{d\eta^2} \quad ; \quad \frac{\partial^3 \psi}{\partial n^3} = \frac{\partial^2 u}{\partial n^2} = \frac{UA^2}{s^{2k}}\frac{d^3F}{d\eta^3} \tag{Bjd11}$$

and substituting these expressions into equation (Bjd4) and rearranging yields

$$kU^2 F\frac{d^2F}{d\eta^2} \;=\; -\frac{\nu UA^2}{s^{2k-1}}\frac{d^3F}{d\eta^3} \tag{Bjd12}$$

This is the critical point at which we recognize that the similarity solution will work in this case if $k = 1/2$ and the governing equation (Bjd12) can be written with only F and η and without any stray s or n as

$$\left(\frac{U}{2\nu A^2}\right) F\frac{d^2F}{d\eta^2} + \frac{d^3F}{d\eta^3} = 0 \tag{Bjd13}$$

We still have the flexibility to choose a value for A and we choose $A = (U/4\nu)^{\frac{1}{2}}$ so that the governing equation (Bjd13) becomes

$$2F\frac{d^2F}{d\eta^2} + \frac{d^3F}{d\eta^3} = 0 \tag{Bjd13}$$

From this it is evident that F will be a function of η alone provided the boundary conditions can be written in terms of F and η alone. Indeed the boundary conditions listed above become

1. $v = 0$ on the solid surface, $\eta = 0$, and therefore $(dF/d\eta)_{\eta=0} = 0$
2. $u = 0$ on the solid surface, $\eta = 0$, and therefore $F_{\eta=0} = 0$

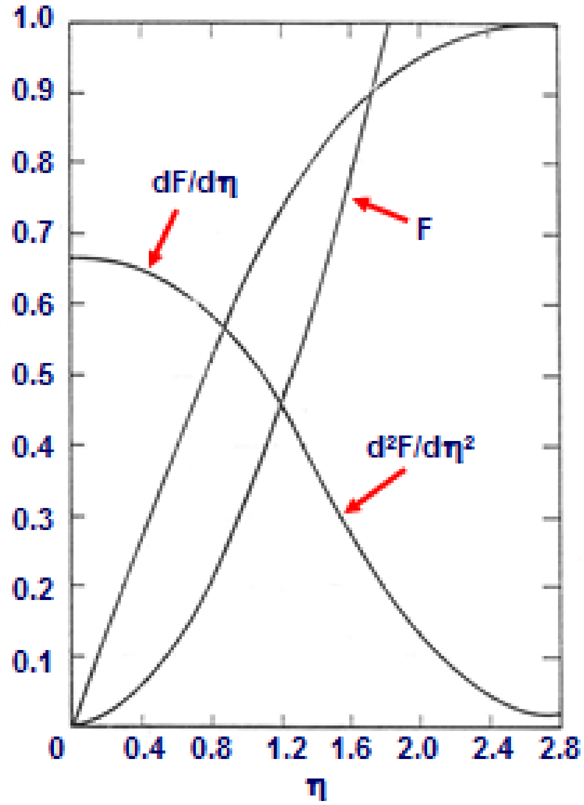

Figure 2: Blasius solution showing the forms of $F(\eta)$, $dF/d\eta$ and $d^2F/d\eta^2$.

3. $u \rightarrow U$ as $\eta \rightarrow \infty$ and therefore $(dF/d\eta)_{\eta\rightarrow\infty} \rightarrow 1$

The absence of any parameters in either the boundary conditions or the governing equation (Bjd13) (resulting from the judicious choice of A) mean that equation (Bjd13) need only be solved once and this is a relatively simple numerical task for which a Runge-Kutta integration coupled with shooting algorithm is suitable. The solution is shown in Figure 2.

In conclusion the Blasius solution for a steady, planar, laminar boundary layer with zero pressure gradient $(dU/ds = 0)$ is

$$\psi = (4\nu U s)^{\frac{1}{2}} F(\eta) \quad \text{where} \quad \eta = \left(\frac{U}{4\nu s}\right)^{\frac{1}{2}} n \tag{Bjd14}$$

and the function $F(\eta)$ is given in Figure 2. This yields velocities, $u(s,n)$ and $v(s,n)$ given by

$$u = U\frac{dF}{d\eta} \quad \text{and} \quad v = \left(\frac{\nu U}{s}\right)^{\frac{1}{2}} \left(F - \eta\frac{dF}{d\eta}\right) \tag{Bjd15}$$

In particular, we note that the magnitude of the velocity ratio, v/u, is given by $(Re_s)^{-\frac{1}{2}}$ where $Re_s = Us/\nu$ is the Reynolds number based on U and the distance, s, along the surface measured from the leading edge. Our initial condition for the boundary layer analysis was that v/u or the inclination of the velocity vector to the solid surface had to be small and we now recognize that this requires Re_s to be large. This will be the case provided $s \gg \nu/U$. Therefore the boundary layer approximation is valid except in a small region close to the leading edge whose length is ν/U. To evaluate this for a typical engineering application involving water at normal temperatures with a kinematic viscosity of $\nu = 10^{-6} m^2/sec$, we note that the length of the region near the leading edge would be $1\mu m$ for a flow with $U = 1\ m/s$. Therefore it is often adequate to neglect this very small region in the flow and to assume it has negligible effect on the flow further downstream.

The above solution for a steady, planar, laminar boundary layer will be used in the sections that follow to illustrate various properties of laminar boundary layers.