Isentropic Duct Flows

In this section we examine the behavior of isentropic flows, continuing the development of the relations in section (Bob). First it is important to identify the set of basic relations that will be used for isentropic, reversible flow and those that are relevant to non-isentropic, irreversible flows that will be the subject of the next section (Bof). In the latter case, one-dimensional flows are governed by equations for

- *Continuity*
- *Momentum*
- *Energy*
- *State*

and these are to be solved for the four unknowns, u, p, ρ and T with specific versions of the equations described in section (Bof). However, with steady, reversible, isentropic flows, the momentum and energy equations have already been used in section (Bob) to derive the isentropic relations and hence the simpler set of four equations that will be further utilized in this section to document the steady, isentropic flow of velocity, u, pressure, p, temperature, T, and density, ρ, in a duct of cross-sectional area, $A(s)$, are

- *Continuity*: $\rho u A = \text{constant}$
- *Energy*: total enthalpy, $h^* = \text{constant}$ which becomes $c_p T + u^2/2 = \text{constant}$
- *State*: $p = \rho \mathcal{R} T$
- *Isentropic Relation*: $p\rho^{-\gamma} \;=\; \text{constant}$

Reworked using the definition of the speed of sound and the Mach number, $M = u/c$, this set of equations can be written as

$$\text{Continuity:} \qquad \rho u A \;=\; \text{constant} \tag{Boe1}$$

$$\text{Energy:} \qquad T\left\{1 + \frac{(\gamma-1)}{2}M^2\right\} \;=\; \text{constant} \;=\; T_0 \tag{Boe2}$$

State and Isentropic Relations:

$$p\left\{1 + \frac{(\gamma-1)}{2}M^2\right\}^{\gamma/(\gamma-1)} \;=\; \text{constant} \;=\; p_0 \tag{Boe3}$$

$$\rho\left\{1 + \frac{(\gamma-1)}{2}M^2\right\}^{1/(\gamma-1)} \;=\; \text{constant} \;=\; \rho_0 \tag{Boe4}$$

where T_0, p_0 and ρ_0 are called *stagnation* or *reservoir* reference quantities since they pertain to conditions where the velocity is zero.

To further develop these relations we write the continuity equation (Boe1) as

$$\rho u A \;=\; \rho M c A \;=\; \rho M (\gamma \mathcal{R} T)^{\frac{1}{2}} A \;=\; \text{constant} \tag{Boe5}$$

and therefore, using the energy equation (Boe2) to substitute for T,

$$\frac{1}{AM}\left[1 + \frac{(\gamma-1)}{2}M^2\right]^{(\gamma+1)/2(\gamma-1)} \;=\; \text{constant} \tag{Boe6}$$

Thus we can compare two locations along the duct denoted by the subscripts 1 and 2 to write

$$\frac{A_1}{A_2} = \frac{M_2}{M_1}\left[\frac{1+(\gamma-1)M_1^2/2}{1+(\gamma-1)M_2^2/2}\right]^{(\gamma+1)/2(\gamma-1)} \tag{Boe7}$$

which directly relates the areas and Mach numbers of the flows at those two locations.

We have already introduced one reference state, namely the *stagnation* or *reservoir* state and, in assessing or calculating isentropic duct flows, it is often convenient to evaluate that reference state as a part of the calculation whether or not that particular state actually occurs in the flow. In other words we evaluate T_0, p_0 and ρ_0 whether or not there is a stagnation point or reservoir in the flow. Furthermore, there is a second reference state which is useful to establish namely the state at the point where the Mach number is unity and the flow is sonic. We shall later discover the physical relevance of this possible location or reference point. For the present, we only need establish that if there is a location within the duct where $M = 1$ then the pressure, temperature, density and area at that location are denoted by p^*, T^*, ρ^* and A^* respectively. For reasons that will emerge later this state is called the *throat* reference state and the conditions at that location are termed the *throat* pressure, temperature and density. By setting $M = 1$ in equations (Boe2) to (Boe4) the throat conditions can be established as

$$\frac{p^*}{p_0} = \left\{\frac{2}{(\gamma+1)}\right\}^{\gamma/(\gamma-1)} \quad ; \quad \frac{T^*}{T_0} = \left\{\frac{2}{(\gamma+1)}\right\} \quad ; \quad \frac{\rho^*}{\rho_0} = \left\{\frac{2}{(\gamma+1)}\right\}^{1/(\gamma-1)} \tag{Boe8}$$

and for air with $\gamma = 1.4$ these yield $p^*/p_0 = 0.528$, $T^*/T_0 = 0.634$ and $\rho^*/\rho_0 = 0.833$. Furthermore, it is often useful to relate the conditions at some arbitrary location given by a Mach number, M, to the throat conditions and the relations for this purpose also follow from equations (Boe2) to (Boe4) and (Boe7) as

$$\frac{T}{T^*} = \left[\frac{(\gamma+1)}{2}\right]\left[1+\frac{(\gamma-1)}{2}M^2\right]^{-1} \tag{Boe9}$$

$$\frac{p}{p^*} = \left[\frac{(\gamma+1)}{2}\right]^{\gamma/(\gamma-1)}\left[1+\frac{(\gamma-1)}{2}M^2\right]^{-\gamma/(\gamma-1)} \tag{Boe10}$$

$$\frac{\rho}{\rho^*} = \left[\frac{(\gamma+1)}{2}\right]^{1/(\gamma-1)}\left[1+\frac{(\gamma-1)}{2}M^2\right]^{-1/(\gamma-1)} \tag{Boe11}$$

$$\frac{A}{A^*} = M^{-1}\left[1+\frac{(\gamma-1)}{2}M^2\right]^{(\gamma+1)/2(\gamma-1)}\left[\frac{\gamma+1}{2}\right]^{-(\gamma+1)/2(\gamma-1)} \tag{Boe12}$$

$$\frac{u}{u^*} = \left\{\frac{\rho^*}{\rho}\right\}\left\{\frac{A^*}{A}\right\} \tag{Boe13}$$

where the last result follows from the continuity equations and equations (Boe11) and (Boe12).

In Figures 1 and 2 the quantities A/A^*, T/T_0, p/p_0, ρ/ρ_0, T/T^*, p/p^*, ρ/ρ^*, and u/u^* are tabulated against M for air ($\gamma = 1.4$, $\mathcal{R} = 280\ m^2/s^2K^o$). The values of T/T_0, p/p_0, and ρ/ρ_0 are also plotted against M in Figure 3 and the value of A/A^* is plotted in Figure 4.

M	A/A*	T/T_o	p/p_o	ρ/ρ_o	T/T*	p/p*	ρ/ρ^*	u/u*
0		1	1	1	1.2	1.8929	1.5774	0
0.05	11.5914	0.9995	0.9983	0.9988	1.1994	1.8896	1.5755	0.0548
0.1	5.82183	0.998	0.993	0.995	1.1976	1.8797	1.5696	0.1094
0.15	3.91034	0.9955	0.9844	0.9888	1.1946	1.8634	1.5598	0.1639
0.2	2.96352	0.9921	0.9725	0.9803	1.1905	1.8409	1.5463	0.2182
0.25	2.40271	0.9877	0.9575	0.9694	1.1852	1.8124	1.5292	0.2722
0.3	2.03507	0.9823	0.9395	0.9564	1.1788	1.7783	1.5086	0.3257
0.35	1.77797	0.9761	0.9188	0.9413	1.1713	1.7392	1.4848	0.3788
0.4	1.59014	0.969	0.8956	0.9243	1.1628	1.6953	1.458	0.4313
0.45	1.44867	0.9611	0.8703	0.9055	1.1533	1.6474	1.4284	0.4833
0.5	1.33984	0.9524	0.843	0.8852	1.1429	1.5958	1.3963	0.5345
0.55	1.25495	0.943	0.8142	0.8634	1.1315	1.5412	1.362	0.5851
0.6	1.1882	0.9328	0.784	0.8405	1.1194	1.4841	1.3258	0.6348
0.65	1.13562	0.9221	0.7528	0.8164	1.1065	1.4251	1.2879	0.6837
0.7	1.09437	0.9107	0.7209	0.7916	1.0929	1.3647	1.2487	0.7318
0.75	1.06242	0.8989	0.6886	0.766	1.0787	1.3034	1.2084	0.7789
0.8	1.03823	0.8865	0.656	0.74	1.0638	1.2418	1.1673	0.8251
0.85	1.02067	0.8737	0.6235	0.7136	1.0485	1.1803	1.1257	0.8704
0.9	1.00886	0.8606	0.5913	0.687	1.0327	1.1192	1.0838	0.9146
0.95	1.00215	0.8471	0.5595	0.6604	1.0165	1.059	1.0418	0.9578
1	1	0.8333	0.5283	0.6339	1	1	1	1
1.05	1.00203	0.8193	0.4979	0.6077	0.9832	0.9424	0.9585	1.0411
1.1	1.00793	0.8052	0.4684	0.5817	0.9662	0.8866	0.9176	1.0812
1.15	1.01745	0.7908	0.4398	0.5562	0.949	0.8326	0.8773	1.1203
1.2	1.03044	0.7764	0.4124	0.5311	0.9317	0.7806	0.8378	1.1583
1.25	1.04675	0.7619	0.3861	0.5067	0.9143	0.7308	0.7993	1.1952
1.3	1.0663	0.7474	0.3609	0.4829	0.8969	0.6832	0.7618	1.2311
1.35	1.08904	0.7329	0.337	0.4598	0.8794	0.6379	0.7253	1.266
1.4	1.11493	0.7184	0.3142	0.4374	0.8621	0.5948	0.69	1.2999
1.45	1.14396	0.704	0.2927	0.4158	0.8448	0.5541	0.6559	1.3327
1.5	1.17617	0.6897	0.2724	0.395	0.8276	0.5156	0.6231	1.3646
1.55	1.21157	0.6754	0.2533	0.375	0.8105	0.4794	0.5915	1.3955
1.6	1.25024	0.6614	0.2353	0.3557	0.7937	0.4454	0.5611	1.4254
1.65	1.29222	0.6475	0.2184	0.3373	0.777	0.4134	0.5321	1.4544
1.7	1.33761	0.6337	0.2026	0.3197	0.7605	0.3835	0.5043	1.4825
1.75	1.38649	0.6202	0.1878	0.3029	0.7442	0.3555	0.4778	1.5097
1.8	1.43898	0.6068	0.174	0.2868	0.7282	0.3294	0.4524	1.536
1.85	1.49519	0.5936	0.1612	0.2715	0.7124	0.3051	0.4283	1.5614

Figure 1: Table of ratios for isentropic duct flows.

M	A/A*	T/T$_o$	p/p$_o$	ρ/ρ$_o$	T/T*	p/p*	ρ/ρ*	u/u*
1.9	1.55526	0.5807	0.1492	0.257	0.6969	0.2825	0.4054	1.5861
1.95	1.61931	0.568	0.1381	0.2432	0.6816	0.2615	0.3836	1.6099
2	1.6875	0.5556	0.1278	0.23	0.6667	0.2419	0.3629	1.633
2.05	1.75999	0.5433	0.1182	0.2176	0.652	0.2238	0.3433	1.6553
2.1	1.83694	0.5313	0.1094	0.2058	0.6376	0.207	0.3246	1.6769
2.15	1.91854	0.5196	0.1011	0.1946	0.6235	0.1914	0.307	1.6977
2.2	2.00497	0.5081	0.0935	0.1841	0.6098	0.177	0.2903	1.7179
2.25	2.09644	0.4969	0.0865	0.174	0.5963	0.1637	0.2745	1.7374
2.3	2.19313	0.4859	0.08	0.1646	0.5831	0.1514	0.2596	1.7563
2.35	2.29528	0.4752	0.074	0.1556	0.5702	0.14	0.2455	1.7745
2.4	2.4031	0.4647	0.0684	0.1472	0.5576	0.1295	0.2322	1.7922
2.45	2.51683	0.4544	0.0633	0.1392	0.5453	0.1198	0.2196	1.8092
2.5	2.63672	0.4444	0.0585	0.1317	0.5333	0.1108	0.2077	1.8257
2.55	2.76301	0.4347	0.0542	0.1246	0.5216	0.1025	0.1965	1.8417
2.6	2.89598	0.4252	0.0501	0.1179	0.5102	0.0949	0.1859	1.8571
2.65	3.03588	0.4159	0.0464	0.1115	0.4991	0.0878	0.176	1.8721
2.7	3.18301	0.4068	0.043	0.1056	0.4882	0.0813	0.1665	1.8865
2.75	3.33766	0.398	0.0398	0.0999	0.4776	0.0753	0.1576	1.9005
2.8	3.50012	0.3894	0.0368	0.0946	0.4673	0.0698	0.1493	1.914
2.85	3.67072	0.381	0.0341	0.0896	0.4572	0.0646	0.1414	1.9271
2.9	3.84977	0.3729	0.0317	0.0849	0.4474	0.0599	0.1339	1.9398
2.95	4.0376	0.3649	0.0293	0.0804	0.4379	0.0556	0.1269	1.9521
3	4.23457	0.3571	0.0272	0.0762	0.4286	0.0515	0.1202	1.964
3.5	6.78962	0.2899	0.0131	0.0452	0.3478	0.0248	0.0714	2.0642
4	10.7188	0.2381	0.0066	0.0277	0.2857	0.0125	0.0436	2.1381
4.5	16.5622	0.198	0.0035	0.0174	0.2376	0.0065	0.0275	2.1936
5	25	0.1667	0.0019	0.0113	0.2	0.0036	0.0179	2.2361
5.5	36.869	0.1418	0.0011	0.0076	0.1702	0.002	0.012	2.2691
6	53.1798	0.122	0.0006	0.0052	0.1463	0.0012	0.0082	2.2953
6.5	75.1343	0.1058	0.0004	0.0036	0.127	0.0007	0.0057	2.3163
7	104.143	0.0926	0.0002	0.0026	0.1111	0.0005	0.0041	2.3333
7.5	141.841	0.0816	0.0002	0.0019	0.098	0.0003	0.003	2.3474
8	190.109	0.0725	0.0001	0.0014	0.087	0.0002	0.0022	2.3591
8.5	251.086	0.0647	7E-05	0.0011	0.0777	0.0001	0.0017	2.3689
9	327.189	0.0581	5E-05	0.0008	0.0698	9E-05	0.0013	2.3772
9.5	421.131	0.0525	3E-05	0.0006	0.063	6E-05	0.001	2.3843
10	535.938	0.0476	2E-05	0.0005	0.0571	4E-05	0.0008	2.3905

Figure 2: Table of ratios for isentropic duct flows.

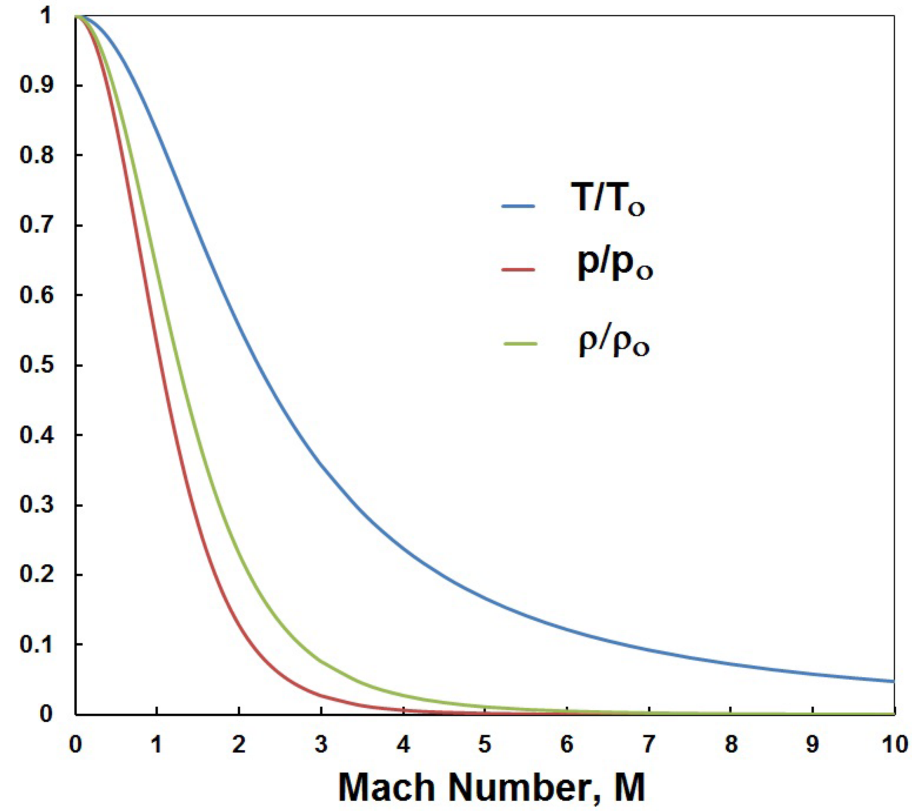

Figure 3: Graphs of T/T_0, p/p_0, and ρ/ρ_0 against Mach Number, M.

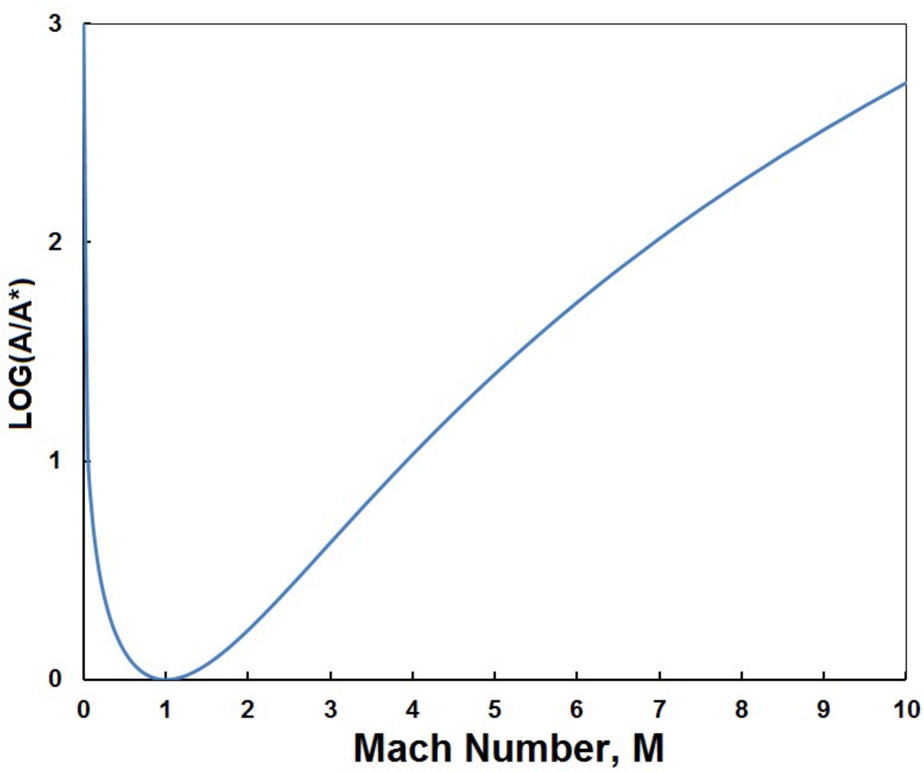

Figure 4: Graph of A/A^* against Mach Number, M.

There are several important properties of the data in Figures 1, 2, 3 and 4 that need to be identified and analyzed. Note first that the tables and graphs cover both subsonic ($M < 1$) and supersonic ($M > 1$) flow and, in this analysis, it will be convenient to use the name *nozzle* for a duct whose cross-sectional area, A, is decreasing in the direction of flow and the name *diffuser* for one whose area is increasing in the direction of flow. Then the data exhibit the following trends for the four options illustrated in Figure

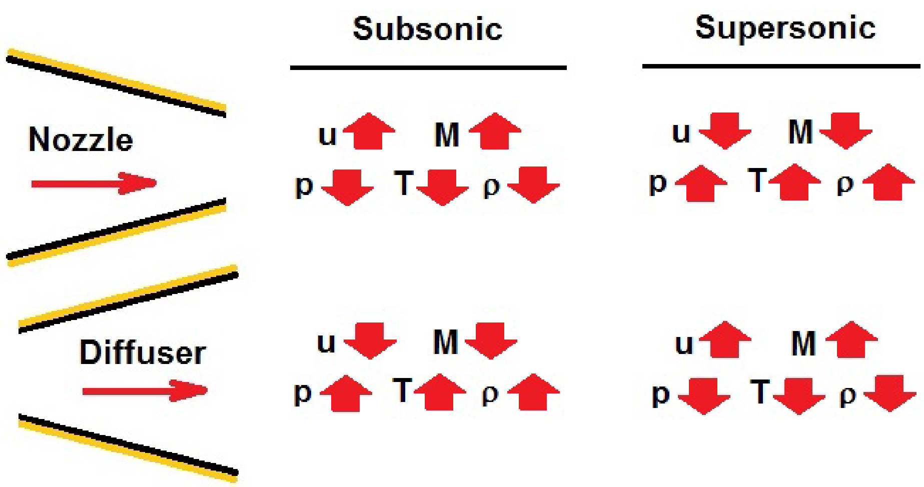

Figure 5: Variations of flow properties in a nozzle and a diffuser.

5: Subsonic flow in a nozzle will feature a flow in which the velocity, u, and Mach number, M, increase in the direction of flow while the pressure, temperature and density decrease. In contrast supersonic flow in a nozzle will feature a velocity, u, and Mach number, M, that decrease in the direction of flow while the pressure, temperature and density increase. [Perhaps it seems strange to envisage a nozzle in which the velocity is decreasing in the direction of the flow because conservation of mass appears to be violated but, in fact, mass conservation is satisfied because the density is increasing faster than the velocity is decreasing.] As depicted by the matrix in Figure 5, a diffuser will exhibit trends that are the reverse of those in a nozzle.

But the trends depicted in Figure 5 raise another set of questions. What happens in a subsonic nozzle flow when the Mach number reaches unity and the area continues to decrease? What happens in a supersonic diffuser as the area continues to increase? Does the flow continue to accelerate indefinitely? We will address the first issue in the next section and the other issues after we have identified and analyzed the phenomenon of a shock wave.