Prandtl-Glauert Mapping

Turning now to subsonic flow and the elliptic partial differential equation (Bon9) that governs such a flow, namely

$$(1-M^2)\frac{\partial^2\phi}{\partial x_1^2}+\frac{\partial^2\phi}{\partial x_2^2}+\frac{\partial^2\phi}{\partial x_3^2} \;=\; 0 \tag{Bop1}$$

the Prandtl-Glauert transformation from the x_i coordinate system to the following x_i^* coordinate system, namely

$$x_1^* \;=\; x_1 \quad ; \quad x_2^* \;=\; (1-M^2)^{\frac{1}{2}}x_2 \quad ; \quad x_3^* \;=\; (1-M^2)^{\frac{1}{2}}x_3 \tag{Bop2}$$

transforms the governing equation (Bop1) to Laplace's equation

$$\frac{\partial^2\phi}{\partial x_1^2}+\frac{\partial^2\phi}{\partial x_2^2}+\frac{\partial^2\phi}{\partial x_3^2} \;=\; 0 \tag{Bop3}$$

Moreover if we define a modified velocity potential, ϕ^*, such that

$$\phi^* \;=\; (1-M^2)^{\frac{1}{2}}\phi \tag{Bop4}$$

then the governing equation is still Laplace's equation:

$$\frac{\partial^2\phi^*}{\partial x_1^2}+\frac{\partial^2\phi^*}{\partial x_2^2}+\frac{\partial^2\phi^*}{\partial x_3^2} \;=\; 0 \tag{Bop5}$$

and, in addition, the appropriate boundary conditions for a slender body become identical to those for ϕ in the original coordinate system. For example, for planar flow, the boundary condition (Bon11),

$$\left(\frac{\partial\phi}{\partial x_2}\right)_{x_2=0} \;=\; U\left(\frac{\partial h}{\partial x_1}\right) \qquad \text{becomes} \qquad \left(\frac{\partial\phi^*}{\partial x_2^*}\right)_{x_2^*=0} \;=\; U\left(\frac{\partial h}{\partial x_1^*}\right) \tag{Bop6}$$

Consequently, with the Prandtl-Glauert transformation, the extensive methods of and solutions for incompressible potential flow can be transformed into solutions for subsonic compressible flow.

To illustrate this further, suppose we begin with an incompressible planar flow solution (for example the flow past a Joukowski airfoil, section (Bged)) denoted by ϕ^*. Then, for a particular Mach number, M, we could compute the velocity potential, ϕ, for that subsonic compressible potential flow, ϕ, around the same object. The known coefficient of pressure for the incompressible flow, C_p^*, can then be used to calculate the coefficient of pressure, C_p, in the compressible subsonic flow using

$$C_p \;=\; \frac{1}{(1-M^2)^{\frac{1}{2}}}C_p^* \tag{Bop7}$$

Similarly the coefficients of lift and drag, C_L and C_D, for the compressible subsonic flow will follow from the known coefficients, C_L^* and C_D^*, for the incompressible flow:

$$C_L \;=\; \frac{1}{(1-M^2)^{\frac{1}{2}}}C_L^* \qquad \text{and} \qquad C_D \;=\; \frac{1}{(1-M^2)^{\frac{1}{2}}}C_D^* \tag{Bop8}$$

though the latter is usually zero in potential flow. We note that the lift and drag coefficients tend to increase as the Mach number increases in such subsonic flows (see section (Bok)). Figure 1 shows how the drag coefficients around some typical objects in subsonic compressible flow tend to increase as the subsonic Mach number is increased (the behavior in transonic flow is less readily explained though the decrease in supersonic flow was anticipated in earlier sections.)

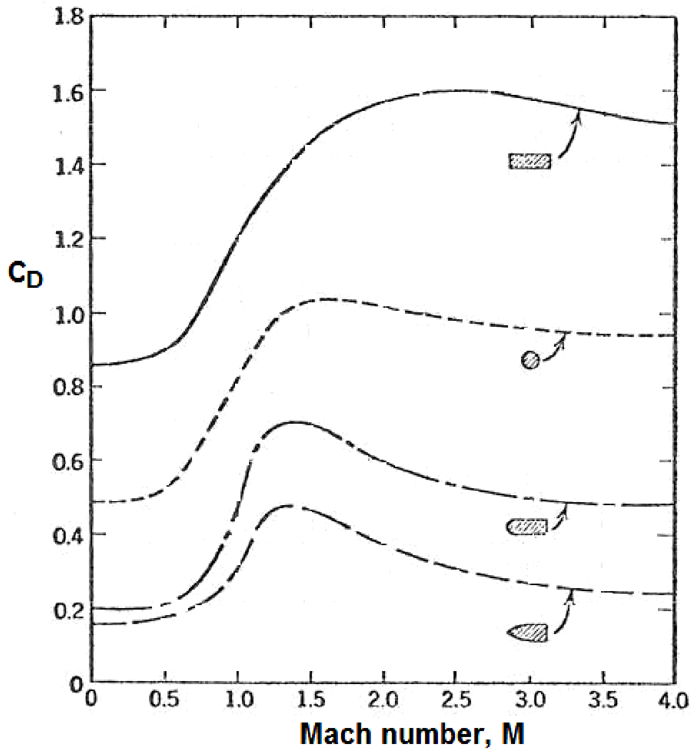

Figure 1: The coefficient of drag based on frontal projected area as a function of Mach number, $M = U/c$, for four different projectile shapes as shown.