Speed of Sound

The next step in our treatment of compressible flows is to develop relations for and understanding of the speed of sound in a compressible fluid. Sound consists of small amplitude pressure waves propagating through the fluid and, in most of the analysis which follows we will retain only the mathematical terms that are linear in the pressure perturbations while neglecting all terms that are quadratic or higher order in those perturbations. We begin with the following thought experiment. Consider a rigid tube of cross-sectional area, A, that contains a piston that is initially at rest. The tube is filled with gas at rest with pressure, p, temperature, T, and density, ρ, but at time $t = 0$ it is suddenly given a small velocity, Δu, directed into the gas as sketched in Figure 1. This generates a wave that travels down the pipe with velocity, V, and

Figure 1: Tube with piston set in motion at $t = 0$, with small velocity, Δu.

leaves behind it gas that now has a velocity, Δu, and increased pressure, $p + \Delta p$, temperature, $T + \Delta T$, and density, $\rho + \Delta\rho$. We seek to determine the wave velocity, V, by applying the basic conservation laws

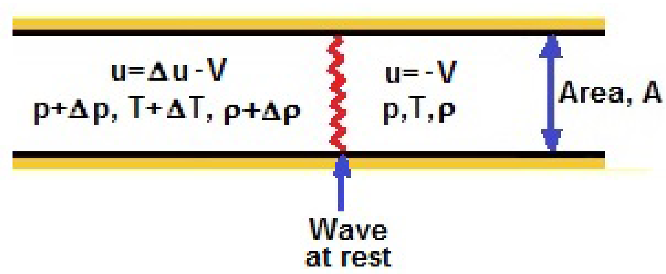

Figure 2: Flow with wave at rest.

that apply to the gas flow. To do so we examine the flow in a frame fixed in the wave as sketched in Figure 2 by applying a Galilean transformation to the flow in Figure 1. Conservation of mass across the wave then requires that

$$A(\rho + \Delta\rho)(\Delta u - V) \;=\; -\rho V A \qquad \text{(Bod1)}$$

so that

$$\Delta u \;=\; \frac{V\Delta\rho}{(\rho + \Delta\rho)} \qquad \text{(Bod2)}$$

In addition, applying the momentum equation to the wave yields

$$A\Delta p \;=\; A\rho V^2 - A(\rho + \Delta\rho)(\Delta u - V)^2 \qquad \text{(Bod3)}$$

that, using the continuity result (Bod2), leads to

$$V^2 \;=\; \frac{\Delta p}{\Delta\rho}\left\{1 + \frac{\Delta\rho}{\rho}\right\} \qquad \text{(Bod4)}$$

The speed of sound is defined as the speed of the wave for very small perturbations or

$$V \to c \qquad \text{when} \qquad \Delta u \to 0 \qquad (\text{and so} \quad \Delta p, \Delta T, \Delta\rho \to 0) \tag{Bod5}$$

and therefore

$$c^2 \;=\; \frac{dp}{d\rho} \tag{Bod6}$$

This result holds for any substance whether solid, liquid or gas and, at various points throughout this book we will use it for many different substances. However, there is one issue with it. As described in the thermodynamic preliminaries, the state of a substance is defined by any two thermodynamic variables. Consequently the left hand side of the relation (Bod6) is incompletely defined; it is necessary to specify the thermodynamic quantity that is being held fixed as the derivative $dp/d\rho$ is being determined. The most common additional assumption is that the gas undergoes an adiabatic change as it passes through the wave and this leads to the adiabatic speed of sound, c_A, which is the speed that will be used throughout this book unless otherwise specified. However, there are circumstances in which the isothermal speed of sound, c_T, is relevant.

In most of the sections which follow we will employ the perfect gas law. The isentropic relations for a perfect gas lead to the following expressions for the isothermal and adiabatic (or isentropic) speeds of sound:

- *Isothermal Speed of Sound*, $c_T \;=\; (\mathcal{R}T)^{\frac{1}{2}}$
- *Adiabatic Speed of Sound*, $c_A \;=\; (\gamma\mathcal{R}T)^{\frac{1}{2}}$

We note that in air c_A is 18.3% larger than c_T.

It is appropriate to detail some examples of the magnitude of the speed of sound. In air with $\gamma = 1.4$ and $\mathcal{R} = 280\ m^2/s^2\ K^o$, the adiabatic speed of sound at a temperature of $T = 293^oK$ is $339m/s$. In terms familiar to airline passengers this is $1220kph$ or $758mph$ though at higher altitudes where the temperature is less, c is significantly smaller. In a liquid whose compressibility is usually quoted as the *adiabatic bulk modulus* (see section (Abc)) denoted by κ and defined by

$$\kappa \;=\; \frac{\rho}{\{d\rho/dp\}_A} \tag{Bod7}$$

it follows that

$$c_A \;=\; \{\kappa/\rho\}^{\frac{1}{2}} \tag{Bod8}$$

For example, in water with an adiabatic bulk modulus of $\kappa = 2.24 \times 10^9\ kg/ms^2$ and a density $\rho = 1000\ kg/m^3$, the adiabatic speed of sound is $c_A = 1500\ m/s$ (section (Abc)), very much greater than the speed of sound in air at normal temperatures.

Before leaving this section it is valuable to observe how the speed of the wave differs from the speed of sound when the amplitude of the wave is not so small (the wave is no longer a sound wave). The result in equation (Bod4) can be approximated by

$$V^2 \;\approx\; c^2\left\{1 + \frac{\Delta\rho}{\rho}\right\} \tag{Bod9}$$

and so the speed of the wave is greater than c when $\Delta\rho$, Δp, and ΔT are positive (they all either increase together or decrease together). Such a wave is called a *compression wave* and, as we will see later, leads to a *shock wave*. It follows that the speed of a shock wave is greater than c. In contrast, the speed of the wave is less than c when $\Delta\rho$, Δp, and ΔT are negative. Such a wave is called an *expansion wave* or *rarefaction wave* and travels at a speed less than the speed of sound. In sections that follow we will explore the consequences of these results.

Finally, we note that the *Mach number*, M, in any flow with velocity, u, is defined as $M = u/c$. Commonly we will refer to the Mach number, $M = U/c$, based on the reference velocity for the flow, U, which is often the free stream velocity ahead of an object or the velocity, for example, of the tip of a compressor blade. But there will also be the need to refer to a *local Mach number* based on the velocity and temperature at a particular location in the flow.

When $M < 1$ at some point in a flow the conditions are termed *subsonic* while if $M = 1$ they are termed *sonic* and if $M > 1$ they are called supersonic. Flows in which $M < 1$ are referred to as *subsonic flows* while those in which $M > 1$ are called *supersonic flows*. There are, of course some flows, where part of the flow is subsonic and other parts are supersonic; such flows are termed *transonic*. The adjective *hypersonic* is normally used for very high speed flows in which the Mach number is greater than about 5.