Constitutive Laws

The constitutive laws for a substance, whether solid or fluid, are the relations between the forces imposed on that substance and the resulting deformation at a macroscopic level. In essence, they represent the collaborative effects of the intermolecular forces when the substance is stretched, twisted, sheared or otherwise deformed. A substance can have many different constitutive laws pertaining to different deformations. For example, Laplace's equation for the relation between the surface tension pressure difference and the curvature of a liquid interface is one such constitutive law as is Fick's law for heat conduction. However, the constitutive law which we will introduce here is that connecting the state of stress within a substance as defined by the stress tensor, σ_{ij} (see section (Bha)), and the displacement or velocity gradients caused by those stresses as defined by the strain tensor, E_{ij}, or the rate of strain tensor, e_{ij} (see section (Bba)). That relationship between the state of stress and the state of strain within a substance is a material property resulting from the intermolecular forces and bonds.

Though the focus here will be on the constitutive laws for a fluid, it is useful to put these in context by also describing some of the basic constitutive laws for a solid. These allow us to again identify the differences between a fluid and a solid and also describe the laws for substances having both fluid and solid properties. In the semi-rigid structure of a solid one might surmise that the obvious constitutive law would be a linear relationship analogous to that between the load and extension of a simple spring; in other words a linear relation between the stress tensor, σ_{ij}, and the strain tensor, E_{ij}. However, such a relation has to be supplemented by the normal stresses, σ_{ii}, that result from uniform isotropic displacement of the material as given by the dilation, Φ^*. Thus the constitutive law for a homogeneous isotropic solid, known as Hooke's Law, is

$$\sigma_{ij} \;=\; \Lambda \delta_{ij} \Phi^* + 2GE_{ij} \qquad \text{(Bhc1)}$$

where δ_{ij} is the Kronecker delta ($\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$). The factors of proportionality, the Lamé constant, Λ, and the shear modulus, G, are related to Young's modulus, E, and Poisson's ratio, Σ, by

$$\Lambda \;=\; \frac{E\Sigma}{(1+\Sigma)(1-2\Sigma)} \qquad \text{and} \qquad G \;=\; \frac{E}{2(1+\Sigma)} \qquad \text{(Bhc2)}$$

While equation (Bhc1) is the most commonly used constitutive law for a simple solid there are many variants representing different solids and their properties.

In contrast, a fluid is defined as a substance in which the stresses do not depend on the displacement but do depend on the velocity gradients within the fluid. The most commonly used constitutive law for a simple fluid, known as a *Newtonian fluid*, is

$$\sigma_{ij} \;=\; \delta_{ij} \left\{ -p + (\Lambda - \frac{2}{3}\mu)\Phi \right\} + 2\mu e_{ij} \qquad \text{(Bhc3)}$$

While the last term involving the viscosity, μ, is the expected linear relation for the stress associated with the shear in the fluid (the factor 2 is included simply so that the stress and shear rate in a simple Couette flow are directly connected by the viscosity) , the first term, which includes the Kronecker delta and therefore only contributes to the normal stresses, needs some further explanation. The first part of the first term, $-p\delta_{ij}$ incorporates the normal stresses that would obviously be associated with a uniform isotropic pressure, p. The second part is another isotropic contribution which vanishes for an incompressible fluid in which $\Phi = 0$ and involves the parameter, Λ, the so-called second coefficient of viscosity. For a monatomic gas $\Lambda = -2\mu/3$.

For an incompressible Newtonian fluid the constitutive law becomes

$$\sigma_{ij} \;=\; -\delta_{ij}p + 2\mu e_{ij} \;=\; -\delta_{ij}p + \mu\left\{\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right\} \tag{Bhc4}$$

and comprises the isotropic component, $-p$, and the *deviatoric* stress, tensor, σ_{ij}^D:

$$\sigma_{ij}^D \;=\; 2\mu e_{ij} \tag{Bhc5}$$

In terms of its Cartesian components, the constitutive law for an incompressible Newtonian fluid is:

$$\sigma_{xx} = -p + 2\mu\frac{\partial u}{\partial x} \quad ; \quad \sigma_{yy} = -p + 2\mu\frac{\partial v}{\partial y} \quad ; \quad \sigma_{zz} = -p + 2\mu\frac{\partial w}{\partial z} \tag{Bhc6}$$

$$\sigma_{xy} = \mu\left\{\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right\} \quad ; \quad \sigma_{yz} = \mu\left\{\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right\} \quad ; \quad \sigma_{zx} = \mu\left\{\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right\} \tag{Bhc7}$$

and the forms of the same constitutive law in cylindrical and in spherical coordinates are included in the adjoining sections (Bhd) and (Bhe).