Equations of Motion in terms of Stress

Having established the stress tensor and therefore all the forces acting on the surface of a differential element, $dxdydz$, the net force due to the general stress configuration can be assessed and substituted into Newton's law. It is most instructive to separately evaluate the net force in each of the Cartesian directions. All of the forces in the x direction are shown in Figure 1 and the sum of these in the positive x direction is

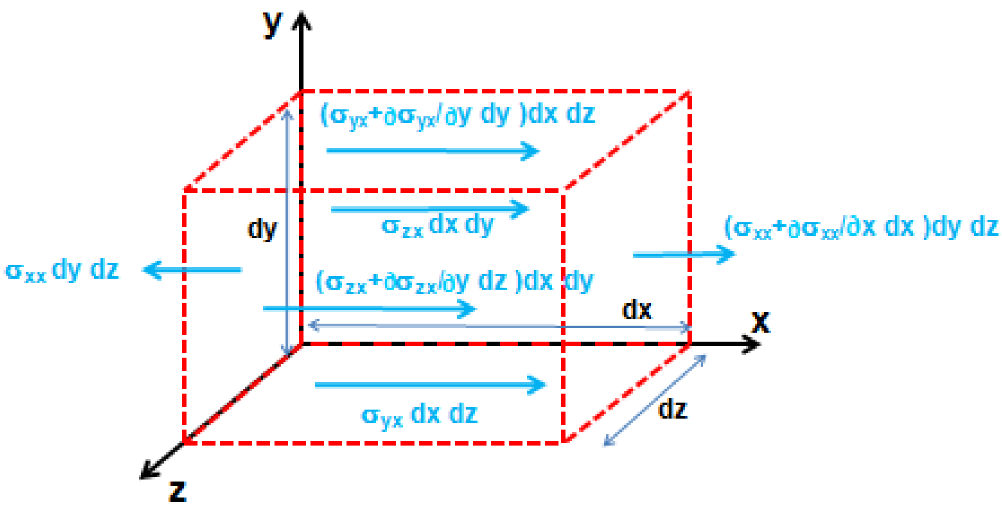

Figure 1: Forces acting in the x direction.

$$\left\{\sigma_{yx}+\frac{\partial\sigma_{yx}}{\partial y}dy\right\}dxdz-\sigma_{yx}dxdz+\left\{\sigma_{zx}+\frac{\partial\sigma_{zx}}{\partial z}dz\right\}dxdy-\sigma_{zx}dxdy+\left\{\sigma_{xx}+\frac{\partial\sigma_{xx}}{\partial x}dx\right\}dydz-\sigma_{xx}dydz \tag{Bhb1}$$

and this becomes

$$\left\{\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{yx}}{\partial y}+\frac{\partial\sigma_{zx}}{\partial z}\right\}dxdydz \tag{Bhb2}$$

A similar analysis of the net force in the y direction yields

$$\left\{\frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{zy}}{\partial z}\right\}dxdydz \tag{Bhb3}$$

and the net force in the z direction yields

$$\left\{\frac{\partial\sigma_{xz}}{\partial x}+\frac{\partial\sigma_{yz}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}\right\}dxdydz \tag{Bhb4}$$

and therefore the net force vector acting on the surfaces of the control volume can be written as

$$\frac{\partial\sigma_{ij}}{\partial x_j}dxdydz \tag{Bhb5}$$

This is to be compared with the net surface force vector obtained during the derivation of Euler's equations and in the absence of viscous shear forces, namely

$$-\frac{\partial p}{\partial x_i}dxdydz \tag{Bhb6}$$

Then, if the expression (Bhb5) instead of the expression (Bhb6) is used in equation 3, section (Bdb) we obtain the general equations of motion for a homogeneous continuum namely:

$$\rho \left\{ \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right\} = \frac{\partial \sigma_{ij}}{\partial x_j} + f_i \tag{Bhb7}$$

This equation is widely applicable. In solid mechanics when the velocities are zero it is known as the **equilibrium equation**. In fluid mechanics it will form the basis of the Navier-Stokes equations once we have established the relationship between the stresses and the velocities.

Written out for each of three Cartesian directions this equation becomes:

$$\rho \left\{ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \right\} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + f_x \tag{Bhb8}$$

$$\rho \left\{ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \right\} = \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + f_y \tag{Bhb9}$$

$$\rho \left\{ \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \right\} = \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + f_z \tag{Bhb10}$$

and, for reference, the equations in cylindrical and spherical coordinates are written out on separate pages.