# The Navier-Stokes Equations

Substituting the expressions for the stresses in terms of the strain rates from the constitutive law for a fluid into the equations of motion we obtain the important Navier-Stokes equations of motion for a fluid. In passing we should also note that the same process using the constitutive law for a solid yields the so-called *equations of equilibrium* for that solid.

Specifically, substituting the constitutive law for a Newtonian fluid, equation (Bhc3), into the equation of motion (Bhb7) yields the Navier-Stokes equations for a Newtonian fluid with dynamic viscosity, $\mu$:

$$\rho\left\{\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right\} = -\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_j}\left\{\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right\}+\frac{\partial}{\partial x_i}\left(\Lambda\frac{\partial u_j}{\partial x_j}\right)+f_i \tag{Bhf1}$$

For an incompressible Newtonian fluid, the Navier-Stokes equations become:

$$\rho\left\{\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right\} = -\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_j}\left\{\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right\}+f_i \tag{Bhf2}$$

For a uniform viscosity and using the equation of continuity for an incompressible fluid, namely,

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{Bhf3}$$

the Navier-Stokes equations (Bhf2) become

$$\rho\frac{Du_i}{Dt} = \rho\left\{\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right\} = -\frac{\partial p}{\partial x_i}+\mu\frac{\partial^2 u_i}{\partial x_j\partial x_j}+f_i \tag{Bhf4}$$

We shall refer to the terms in this equation as follows: the term (or terms) on the left-hand side is the inertial term, the first term on the right-hand side is the pressure term, the second is the viscous term and the last is the body force term. Written out in its Cartesian components the equations are:

$$\rho\frac{Du}{Dt} = \rho\left\{\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right\} = -\frac{\partial p}{\partial x}+\mu\left\{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right\}+f_x \tag{Bhf5}$$

$$\rho\frac{Dv}{Dt} = \rho\left\{\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right\} = -\frac{\partial p}{\partial y}+\mu\left\{\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right\}+f_y \tag{Bhf6}$$

$$\rho\frac{Dw}{Dt} = \rho\left\{\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right\} = -\frac{\partial p}{\partial z}+\mu\left\{\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right\}+f_z \tag{Bhf7}$$

and in the forms that we will frequently use in the pages ahead, the Navier-Stokes equations for planar, incompressible, Newtonian flow in the $x, y$ plane are:

$$\rho\frac{Du}{Dt} = \rho\left\{\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right\} = -\frac{\partial p}{\partial x}+\mu\left\{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right\}+f_x \tag{Bhf8}$$

$$\rho\frac{Dv}{Dt} = \rho\left\{\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right\} = -\frac{\partial p}{\partial y}+\mu\left\{\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right\}+f_y \tag{Bhf9}$$

In these pages, the Navier-Stokes equations will be deployed to explore a wide range of flows. However, the number of exact solutions to these equations is small and mostly for geometrically simple flows. These are described and explored in the sections which follow. The exact solutions are very limited because the non-linear inertial terms present an intractable mathematical barrier in all but the simplest geometries. Therefore we will resort to approximate solutions for many different types of flows and in doing so will have a need to neglect or approximate some of the terms in the equations in order to make progress. In the process we will estimate the magnitudes of some of the terms relative to other terms. In particular we will compare the magnitudes of the viscous and inertial terms and the ratio of these magnitudes leads to an important fluid flow parameter known as the *Reynolds number*, *Re*.

Consider a Newtonian fluid flow with a typical dimension, $L$, and a typical velocity, $U$. Then one might argue that the inertial terms in the equation of motion (Bhf1), (Bhf2) or (Bhf4) would have a typical magnitude of $\rho U^2/L$ and that the viscous terms would have a typical magnitude of $\mu U/L^2$. Therefore the importance of the inertial terms relative to the viscous terms would be given by the ratio known as the *Reynolds number*, *Re*:

$$Re \;=\; \frac{\rho U^2/L}{\mu U/L^2} \;=\; \frac{\rho U L}{\mu} \;=\; \frac{UL}{\nu} \tag{Bhf10}$$

This number is used throughout the subject of fluid mechanics in order to assess the relative importance of the inertial and viscous terms and therefore to try to justify some simplification to the Navier-Stokes equations.

As a first example one might argue that in flows for which $Re >> 1$ the viscous terms would be unimportant relative to the inertial terms and therefore one could leave out the viscous terms. The equations of motion would then revert to Euler's equations (Bdb3) and the methods utilizing those equations can be used. On the other hand one might also argue that in flows for which $Re << 1$ the inertial terms would be unimportant relative to the viscous terms and therefore one might omit the inertial terms in seeking solutions to the flow. This approach leads to a set of equations known as the equations of *creeping flow* or *Stokes' flow* which are detailed elsewhere in sections (Bl).

If these two arguments were foolproof then fluid mechanics would be a much simpler subject and the various fluid flow phenomena would be less interesting. The problem is that the arguments are not uniformly valid and, in both cases, fail in some particular regions of the flow.

First consider the case of $Re >> 1$. While it is true that the viscous terms are much smaller than the inertial terms in most regions of the flow and can therefore be neglected in seeking solutions to the flows in those regions, there are other regions where this is not true and the viscous terms therefore play a significant role. As we will describe in the section on boundary conditions, the flow at the surface of a solid object experiences what is known as the *no-slip condition* so that the velocity of the fluid in contact with the solid surface is equal to that of the solid surface and is therefore zero for a solid object at rest (for simplicity of demonstration we focus on steady flows past a solid object at rest). Therefore the inertial terms (which involve the velocity) are identically zero at that solid surface and are very small close to the solid surface. In contrast the viscous terms are non-zero since they involve the gradients of the velocity but not the velocity itself. Hence near the solid surface there is a special region in which the viscous terms are actually large compared with the inertial terms and must therefore be included in seeking a solution to the flow in that region. These regions of the flow are called *viscous boundary layers* and their study is an important, indeed crucial, component in the study of flows at high Reynolds numbers. They will be described in much more detail in the sections that follow. Further out in the flow, away from the solid surface, the viscous terms are indeed negligible in many cases and that region of the flow can be treated using Euler's equations of motion and the results, such as Bernoulli's equation, that follow from Euler's equations.

Second, consider the complementary case of $Re << 1$. While it is true that the inertial terms are much smaller than the viscous terms in many regions of the flow and can therefore be neglected in seeking solutions to the flows in those regions, there are other regions where this is not true and the inertial terms therefore play a significant role. Specifically consider again the flow of a uniform stream past a solid object. If $Re << 1$ then anywhere close to the solid object the inertial terms will be small relative to the viscous terms and can be neglected in seeking a solution to the flow anywhere near to the solid object; thus the solution in that region would appropriately utilize the equations of Stokes' flow as described above. However, when we consider the flow at a large distance from the object in a region where the flow is close to being that of a uniform stream, we recognize that the second derivatives that comprise the viscous terms could and do become very small and therefore the inertial terms become comparable to and larger than the viscous terms. In that region, far from the object where the inertial terms are significant we must use a modified version of the Stokes' equations, one version of which are the *Oseen equations* described in a section that follows.