Kinematics of Deformation and Motion

In any homogeneous and isotropic continuum, whether fluid or solid, the deformation or motion of any incremental Lagrangian element can be decomposed into four fundamental deformations or motions. In these sections we will identify these fundamental deformations or motions and relate them to the displacements or velocities of the continuum. These relations are purely kinematic in that they do not involve any mechanical or physical properties of the continuum; consequently they are identical for a solid, fluid or any continuum.

First we address the case of a solid where it is convenient to focus on the vector of displacements denoted by X_i that must be measured relative to some initial location or unstressed state of the substance. The four fundamental displacements are translation, rotation, shear (or extension) and dilation. Translation is obviously described by the displacement vector, X_i, and needs little further comment. At the next level of detail we can identify the deformation that is clearly related to the spatial differences in the displacements and therefore to the spatial gradient of the displacement, $\partial X_i/\partial x_j$. This tensor is called the *displacement gradient tensor*. Note that without any displacement gradient there would be no deformation. Moreover, this displacement gradient tensor combines both the deformation and the rotation of the substance. Thus we separate these two displacements into the *strain tensor*, E_{ij}, and the rotation tensor, Ω_{ij}, which are respectively the symmetric and antisymmetric parts of that displacement gradient tensor. The *strain tensor* is

$$E_{ij} \;=\; \frac{1}{2}\left\{\frac{\partial X_i}{\partial x_j}+\frac{\partial X_j}{\partial x_i}\right\} \qquad \text{(Bba1)}$$

and the rotation tensor is

$$\Omega_{ij} \;=\; \frac{1}{2}\left\{\frac{\partial X_i}{\partial x_j}-\frac{\partial X_j}{\partial x_i}\right\} \qquad \text{(Bba2)}$$

Note that

$$E_{ij} \;=\; E_{ji} \qquad \text{and} \qquad \Omega_{ij} \;=\; -\Omega_{ji} \qquad \text{(Bba3)}$$

We refer to the diagonal components in this tensor as the *normal strains* and the off-diagonal components as *shear strains*. Moreover the sum of the normal strains

$$\Phi^* \;=\; E_{11}+E_{22}+E_{33} \qquad \text{(Bba4)}$$

is known as the *dilation* and is a measure of the volumetric expansion of the substance.

In a precisely parallel way the motions of a fluid are characterized by the rates of change with time of the above measures. Thus the velocities, $u_i(x_i,t)$, are just the Lagrangian rates of change with time of the displacements:

$$u_i \;=\; \frac{DX_i}{Dt} \qquad \text{(Bba5)}$$

and the rate of deformation of the fluid is clearly related to the spatial gradients of the velocities, $\partial u_i/\partial x_j$, a tensor that is called the *velocity gradient tensor*. Note that without any velocity gradient there would be no rate of deformation. Moreover, this velocity gradient tensor combines both the rate of deformation and the rate of rotation of the fluid. Thus we separate these by defining the *strain rate tensor*, e_{ij}, and the rate of rotation tensor, ω^*_{ij}, that are respectively the symmetric and antisymmetric parts of that velocity gradient tensor, $\partial u_i/\partial x_j$. Thus the *rate of strain tensor* is

$$e_{ij} \;=\; \frac{1}{2}\left\{\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right\} \qquad \text{(Bba6)}$$

and the rate of rotation tensor is

$$\omega^*_{ij} = \frac{1}{2}\left\{\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right\} \quad \text{(Bba7)}$$

Note that

$$e_{ij} = e_{ji} \qquad \text{and} \qquad \omega^*_{ij} = -\omega^*_{ji} \quad \text{(Bba8)}$$

We refer to the diagonal components in this matrix as the *normal strain rates* and the off-diagonal components as *shear strain rates*. Moreover the sum of the normal strain rates

$$\Phi = e_{11} + e_{22} + e_{33} \quad \text{(Bba9)}$$

is known as the *dilation rate* and is a measure of the rate of volumetric expansion of the fluid.

Written out in full the rate of strain tensor (or matrix) has Cartesian components

$$e_{xx} = \frac{\partial u}{\partial x} \quad ; \quad e_{yy} = \frac{\partial v}{\partial y} \quad ; \quad e_{zz} = \frac{\partial w}{\partial z} \quad \text{(Bba10)}$$

$$e_{xy} = e_{yx} = \frac{1}{2}\left\{\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right\} \quad \text{(Bba11)}$$

$$e_{yz} = e_{zy} = \frac{1}{2}\left\{\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right\} \quad \text{(Bba12)}$$

$$e_{zx} = e_{xz} = \frac{1}{2}\left\{\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right\} \quad \text{(Bba13)}$$

In cylindrical coordinates, (r, θ, z), with velocities u_r, u_θ, u_z in the r, θ, z directions, the strain rate tensor has components:

$$e_{rr} = \frac{\partial u_r}{\partial r} \quad ; \quad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \quad ; \quad e_{zz} = \frac{\partial u_z}{\partial z} \quad \text{(Bba14)}$$

$$e_{r\theta} = e_{\theta r} = \frac{1}{2}\left\{\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right\} \quad \text{(Bba15)}$$

$$e_{\theta z} = e_{z\theta} = \frac{1}{2}\left\{\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}\right\} \quad \text{(Bba16)}$$

$$e_{zr} = e_{rz} = \frac{1}{2}\left\{\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right\} \quad \text{(Bba17)}$$

In spherical coordinates, (r, θ, ϕ), with velocities u_r, u_θ, u_ϕ in the r, θ, ϕ directions, the strain rate tensor has components:

$$e_{rr} = \frac{\partial u_r}{\partial r} \quad ; \quad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \quad ; \quad e_{\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{u_\theta \cot\theta}{r} \quad \text{(Bba18)}$$

$$e_{r\theta} = e_{\theta r} = \frac{1}{2}\left\{r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right\} \quad \text{(Bba19)}$$

$$e_{\theta\phi} = e_{\phi\theta} = \frac{1}{2}\left\{\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{u_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi}\right\} \quad \text{(Bba20)}$$

$$e_{\phi r} = e_{r\phi} = \frac{1}{2}\left\{\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{u_\phi}{r}\right)\right\} \quad \text{(Bba21)}$$

The other key kinematic property is the rate of rotation tensor, ω^*_{ij} given by

$$\omega^*_{ij} \;=\; -\omega^*_{ji} \;=\; \frac{1}{2}\left\{\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right\} \tag{Bba22}$$

Since $\omega^*_{ij} = -\omega^*_{ji}$ this also allows us to define a very important kinematic property of a flow namely the *vorticity*, denoted by ω_k which is just *twice* the rate of rotation tensor ω^*_{ij} so that

$$\omega_k \;=\; -2\omega^*_{ij} \;=\; \frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j} \tag{Bba23}$$

The vorticity is a key characteristic of a flow and will be addressed in many sections of this book.

Written out in terms of its Cartesian components, the vorticity components are

$$\omega_x \;=\; 2\omega^*_{zy} \;=\; -2\omega^*_{yz} \;=\; \left\{\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right\} \tag{Bba24}$$

$$\omega_y \;=\; 2\omega^*_{xz} \;=\; -2\omega^*_{zx} \;=\; \left\{\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right\} \tag{Bba25}$$

$$\omega_z \;=\; 2\omega^*_{yx} \;=\; -2\omega^*_{xy} \;=\; \left\{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right\} \tag{Bba26}$$

In cylindrical coordinates, (r, θ, z), with velocities u_r, u_θ, u_z in the r, θ, z directions, the rotation tensor and vorticity have components:

$$\omega_r \;=\; 2\omega^*_{z\theta} \;=\; -2\omega^*_{\theta z} \;=\; \left\{\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right\} \tag{Bba27}$$

$$\omega_\theta \;=\; 2\omega^*_{rz} \;=\; -2\omega^*_{zr} \;=\; \left\{\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right\} \tag{Bba28}$$

$$\omega_z \;=\; 2\omega^*_{\theta r} \;=\; -2\omega^*_{r\theta} \;=\; \left\{\frac{1}{r}\frac{\partial (r u_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right\} \tag{Bba29}$$

In spherical coordinates, (r, θ, ϕ), with velocities u_r, u_θ, u_ϕ in the r, θ, ϕ directions, the rotation tensor and vorticity have components:

$$\omega_r \;=\; 2\omega^*_{\phi\theta} \;=\; -2\omega^*_{\theta\phi} \;=\; \frac{1}{r^2 \sin\theta}\left\{\frac{\partial (r u_\phi \sin\theta)}{\partial \theta} - \frac{\partial (r u_\theta)}{\partial \phi}\right\} \tag{Bba30}$$

$$\omega_\theta \;=\; 2\omega^*_{r\phi} \;=\; -2\omega^*_{\phi r} \;=\; \frac{1}{r \sin\theta}\left\{\frac{\partial u_r}{\partial \phi} - \frac{\partial (r u_\phi \sin\theta)}{\partial r}\right\} \tag{Bba31}$$

$$\omega_\phi \;=\; 2\omega^*_{\theta r} \;=\; -2\omega^*_{r\theta} \;=\; \frac{1}{r}\left\{\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right\} \tag{Bba32}$$