Integral approach to the continuity equation

The third and last approach to the invocation of the conservation of mass utilizes the general macroscopic, Eulerian control volume depicted in figure 1. This volume is denoted by V and its surface by S. We consider a small vector segment of that surface $\underline{dS}$ where the magnitude of the vector is the scalar area of the segment dS and the direction of the vector is the outward normal to the surface at that point. Note that if $\underline{n}$ is the outward unit normal to the surface at that point then $\underline{dS} = \underline{n}.dS$.

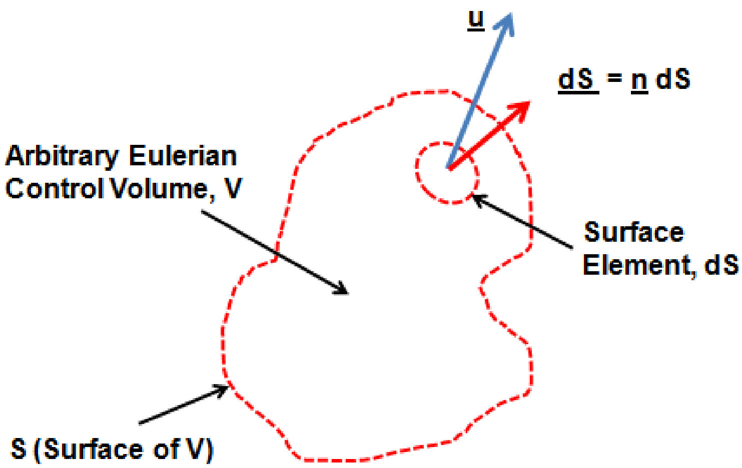

Figure 1: Arbitrary Eulerian control volume.

First we evaluate the net flux of mass *out of* the control volume V. If $\underline{u}$ is the velocity of the fluid at the surface area $\underline{dS}$ then the rate of flow of mass leaving V through that elemental area is

$$\rho\underline{u}.\underline{dS} = (\underline{n}.\underline{u})\rho dS \tag{Bcf1}$$

and integrating over the entire surface the net rate of mass leaving V becomes

$$\int_S \rho\underline{u}.\underline{dS} = \int_S \rho(\underline{n}.\underline{u})\ dS \tag{Bcf2}$$

Invoking the conservation of mass, this must be equal to minus the rate of increase of mass inside V which is

$$-\int_V \frac{\partial\rho}{\partial t}\ dV \tag{Bcf3}$$

so that this integral version of the continuity equation becomes

$$\int_V \frac{\partial\rho}{\partial t}\ dV + \int_S \rho(\underline{n}.\underline{u})\ dS = 0 \tag{Bcf4}$$

It can be seen that this is just a generalization of the macroscopic version of the continuity equation developed earlier.

This integral version of the continuity equation is not only useful in the form given above but is also useful when the last term is converted from a surface integral to a volume integral by using Gauss' theorem. This states that for any general vector quantity, $\underline{q}$,

$$\int_S \underline{q}.\underline{dS} = \int_V \nabla.\underline{q}\ dV \tag{Bcf5}$$

and therefore, since the last term in the integral form of the continuity equation implies $\underline{q} = \rho\underline{u}$ in this instance, that integral continuity equation can be written as

$$\int_V \frac{\partial\rho}{\partial t}\ dV + \int_V \nabla.(\rho\underline{u})\ dV = 0 \tag{Bcf6}$$

or

$$\int_V \frac{\partial\rho}{\partial t} + \nabla.(\rho\underline{u})\ dV = 0 \tag{Bcf7}$$

Since the value of this volume integral is zero and since the choice of the volume V was arbitrary it must follow that the integrand must be everywhere zero and therefore

$$\frac{\partial\rho}{\partial t} + \nabla.(\rho\underline{u}) = 0 \tag{Bcf8}$$

and this is just the differential form of the continuity equation developed earlier. Thus we see that all three approaches lead to the same answer. However, the various forms of the continuity equation developed along the way are all useful in the variety of applications explored in this text.