Continuity equation in other coordinate systems

We recall that in a rectangular Cartesian coordinate system the general continuity equation is

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \quad \text{(Bce1)}$$

or in tensor notation

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_j)}{\partial x_j} = 0 \quad \text{(Bce2)}$$

or in vector notation

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \underline{u}) = 0 \quad \text{(Bce3)}$$

If the flow is planar, the velocity and the derivatives in one direction (say the z-direction) become zero and the continuity equation reduces to

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \quad \text{(Bce4)}$$

and if the flow is incompressible this is further reduced to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{(Bce5)}$$

a form that is repeatedly used in this text.

In a planar flow such as this it is sometimes convenient to use a polar coordinate system (r, θ). Then the continuity equation becomes

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial(\rho r u_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho u_\theta)}{\partial \theta} = 0 \quad \text{(Bce6)}$$

where u_r, u_θ are the velocities in the r and θ directions. If the fluid is incompressible this further reduces to

$$\frac{\partial(r u_r)}{\partial r} + \frac{\partial u_\theta}{\partial \theta} = 0 \quad \text{(Bce7)}$$

and, in the very simple circumstances in which the flow is purely radial so that $u_\theta = 0$, it follows that

$$\frac{\partial(r u_r)}{\partial r} = 0 \quad \text{(Bce8)}$$

and therefore $u_r = C/r$ where C is a constant or only a function of time. This is the flow field for a simple "line source or sink" at $r = 0$ which is perpendicular to the plane of r and θ. We will discuss these flows elsewhere in this book.

Another simple planar flow that is valuable to identify is one in which the vortical flow in which u_r is everywhere zero and the flow proceeds in a circular path. Then the continuity equation for incompressible flow becomes

$$\frac{\partial u_\theta}{\partial \theta} = 0 \quad \text{(Bce9)}$$

and therefore u_θ is a function only of r and t. There are several specific types of vortical motion that are discussed elsewhere in this book.

Reverting to the more general three-dimensional flow, the continuity equation in cylindrical coordinates (r, θ, z) is

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial(\rho r u_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho u_\theta)}{\partial \theta} + \frac{\partial(\rho u_z)}{\partial z} = 0 \tag{Bce10}$$

where u_r, u_θ, u_z are the velocities in the r, θ and z directions of the cylindrical coordinate system. A particular subset of such flows is axisymmetric flow in which the derivatives in the θ direction are zero so that the continuity equation becomes

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial(\rho r u_r)}{\partial r} + \frac{\partial(\rho u_z)}{\partial z} = 0 \tag{Bce11}$$

where u_r, u_θ, u_z are the velocities in the r, θ and z directions of the cylindrical coordinate system. For incompressible axisymmetric flow it follows that the continuity equation becomes

$$\frac{1}{r}\frac{\partial(r u_r)}{\partial r} + \frac{\partial(u_z)}{\partial z} = 0 \tag{Bce12}$$

Another subset is vortical flow about the z axis (with no flow in the z direction) in which case the continuity equation becomes

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial(\rho r u_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho u_\theta)}{\partial \theta} = 0 \tag{Bce13}$$

Turning now to spherical coordinates (r, θ, ϕ) the continuity equation becomes

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial(\rho r^2 u_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\rho u_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial(\rho u_\phi)}{\partial \phi} = 0 \tag{Bce14}$$

and in some particular flows in which this simplifies further. In purely radial flow such as that due to a point source or a sink (as opposed to the line source or sink described above) it becomes

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial(\rho r^2 u_r)}{\partial r} = 0 \tag{Bce15}$$

For incompressible flow this reduces to

$$\frac{\partial(r^2 u_r)}{\partial r} = 0 \tag{Bce16}$$

and therefore $u_r = C/r^2$ where C is only a function of time. Clearly this can be arrived at by arguing that for an incompressible fluid the spherical surface area times the radial velocity must be the same at all radii.