Angular Momentum Theorem

The complement to the linear momentum theorem is the angular momentum theorem which has particular value in addressing various rotating flows. Newton's law of angular motion states that the torque, $\underline{T}$ or T_i, acting on a Lagrangian mass, m, is equal to the (Lagrangian) rate of change of the angular momentum, $(\underline{r} \times m\underline{u})$ where $\underline{r}$ is the position vector of the Lagrangian mass:

$$\underline{T} \;=\; \frac{D}{Dt}\left\{\underline{r} \times m\underline{u}\right\} \tag{Beh1}$$

Alternatively this can be written as

$$\underline{T} \;=\; \underline{r} \times \frac{D}{Dt}\left\{m\underline{u}\right\} + \frac{D\underline{r}}{Dt} \times (m\underline{u}) \;=\; \underline{r} \times \frac{D}{Dt}\left\{m\underline{u}\right\} \tag{Beh2}$$

since $D\underline{r}/Dt = \underline{u}$ and $\underline{u} \times \underline{u} = 0$. It follows that Newton's law of angular motion applied to the Lagrangian volume, V_L, is

$$\underline{T} \;=\; \frac{D}{Dt}\left\{\int_{V_L} \underline{r} \times \rho\underline{u}\; dV_L\right\} \tag{Beh3}$$

and applying the transport theorem to the right hand side of this equation this becomes

$$\underline{T} \;=\; \int_V \frac{\partial(\underline{r} \times \rho\underline{u})}{\partial t} dV \;+\; \int_S (\underline{r} \times \rho\underline{u})\; \underline{u}.\underline{dS} \tag{Beh4}$$

where V is the Eulerian volume coincident with V_L at the moment under consideration. This is the **angular momentum theorem** applied to a fixed Eulerian volume, V. In words it states that the torque acting on any Eulerian volume is equal to the sum of the rate of increase of angular momentum contained within the Eulerian control volume and the flux of angular momentum **out of** the control volume. Note that the last term is evaluated as the mass flow rate out of the control volume, $\rho\underline{u}dS$, multiplied by $\underline{r} \times \underline{u}$ and integrated over the entire surface of the control volume.

This angular momentum theorem will be used, for example, in evaluating the performance of a pump or turbine.