Example: Duct of Arbitrary Shape

As a common and useful example consider the application of the linear momentun theorem to the duct of arbitrary shape depicted in Figure 1. For simplicity, it is assumed that the velocity magnitude, q_1, is uniform across the inlet area, A_1, where the fluid density is ρ_1. Similarly, the velocity magnitude, q_2, is uniform across the outlet area, A_2, where the fluid density is ρ_2. The angular inclination of the outlet flow to the inlet flow is θ. Otherwise the walls of the duct are solid.
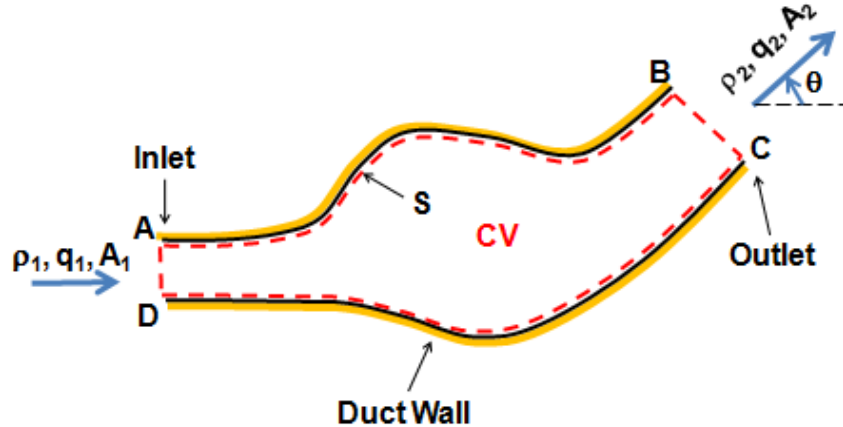

Figure 1: Duct of arbitrary shape.

We will begin by choosing a control volume which follows the inside surface of the duct and cuts across the inlet and outlet perpendicular to the direction of flow in those places. It then follows that the force, F_1, on this control volume in the direction of the q_1 velocity is given by

$$F_1 \;=\; -(\rho_1 q_1 A_1)q_1 + (\rho_2 q_2 A_2)q_2 \cos\theta \qquad \text{(Beb1)}$$

since the component of the outlet velocity in the q_1 direction is $q_2 \cos\theta$. Moreover since continuity requires that the mass flow rate, m, is

$$m \;=\; \rho_1 q_1 A_1 \;=\; \rho_2 q_2 A_2 \qquad \text{(Beb2)}$$

the force in the q_1 direction can be written as

$$F_1 \;=\; m(-q_1 + q_2 \cos\theta) \qquad \text{(Beb3)}$$

and the force, F_2, perpendicular to q_1 but in the plane of q_2 is

$$F_2 \;=\; m(q_2 \sin\theta) \qquad \text{(Beb4)}$$

These are the components of the sum of the forces acting **on** all sides of the control volume, ABCD, which, in this manifestation, contains only fluid. This sum must include both body forces (for example that due to gravity) and the surface forces acting on all sides of the control volume. For example they include the force, $p_1 A_1$, due to the pressure, p_1, acting on the inlet AD and the force, $p_2 A_2$, due to the pressure, p_2, acting on the outlet, BC, (in addition to any viscous forces on those sides) as well as all forces imposed by the walls AB and CD on the fluid touching them.
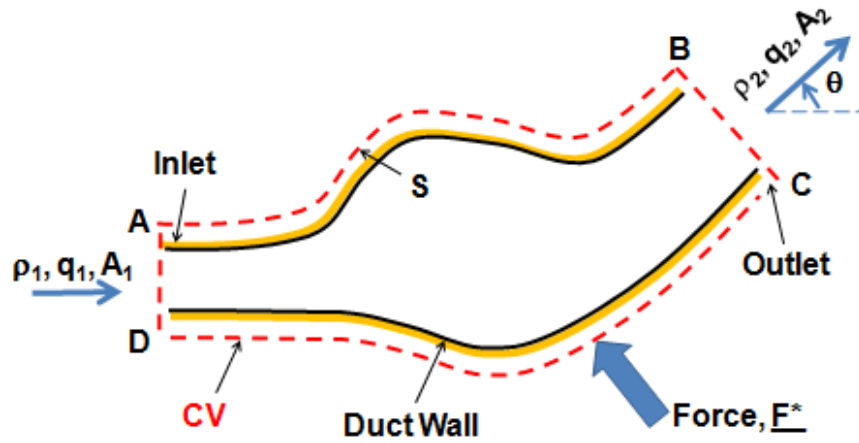
Figure 2: Duct with alternative control volume, CV.

Superficially one might envision difficulties in evaluating the pressure and viscous shear stresses on these walls in order to calculate these last contributions to the force. However, a simple change in the control volume alleviates these concerns. If we choose the control volume shown in Figure 2 in which the sides of the control volume lie outside the walls of the duct, we recognize that the momentum fluxes remain the same (provided the solid structure of the walls remains at rest) and the evaluation of the forces is greatly simplified. At this stage in the analysis, in order to compute the forces due the pressure in this example and all similar problems, we perform one manipulation that clarifies the issue. We denote the pressure acting on the sides of the control volume AB and CD outside the duct by p_a (atmospheric pressure) and assume that this is the same along the length of these exterior surfaces. Since a uniform pressure everywhere on all sides of the control volume would yield no net contribution to the forces, it follows that the only non-zero contributions to the force arise from the pressure differences, $p_1 - p_a$ and $p_2 - p_a$, acting on the inlet and outlet respectively and the force that is required to hold the structure in place (imposed by some structure not shown in the Figures 1 and 2). Indeed, neglecting any viscous forces acting on the inlet and outlet and assuming the structure is not moving or accelerating, the components of the force, $\underline{F^*}$, required to hold the duct in place are then

$$F_1^* = m(-q_1 + q_2 \cos\theta) - (p_1 - p_a)A_1 + (p_2 - p_a)A_2 \cos\theta \qquad \text{(Beb5)}$$

and the force, F_2^*, perpendicular to q_1 but in the plane of q_2 is

$$F_2^* = m(q_2 \sin\theta) + (p_2 - p_a)A_2 \sin\theta \qquad \text{(Beb6)}$$

Thus the forces can be computed given the input and output quantities, m, q_1 q_2, $(p_1 - p_a)$, $(p_2 - p_a)$ as well as the geometric quantities. Of course, those flow quantities will also be related through continuity and by other relations such as Bernoulli's equation. Several worked examples are featured on companion pages.