Linear Momentum Theorem

In previous pages we have used Newton's law of motion applied to an infinitesmal fluid element to develop the differential equations of fluid motion such as Euler's equations and the Navier-Stokes equations. But, just as we found in applying the principle of mass conservation to a flowing fluid, we can also develop very useful tools by considering the application of Newton's law to an macroscopic volume of fluid. To do so consider the arbitrary Lagrangian volume, V_L, which, at the moment of time under consideration, coincides with the Eulerian volume, V, as depicted in Figure 1. Since the Lagrangian volume always contains the
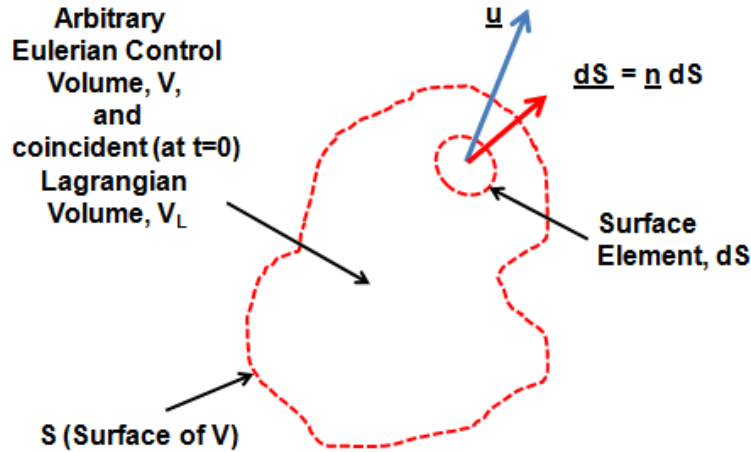

Figure 1: Arbitrary Eulerian control volume, V, and the coincident Lagrangian volume, V_L, at time, $t = 0$.

same mass of fluid (and/or solid), it follows from conservation of mass that

$$\frac{D}{Dt}\left\{\int_{V_L} \rho dV_L\right\} = 0 \tag{Bea1}$$

Newton's law states that the net vector force, $\underline{F}$, acting on this Lagrangian mass must be equal to the Lagrangian rate of change of the vector momentum contained within the volume and since the contained mass remains constant this can be written as

$$\underline{F} = \frac{D}{Dt}\left\{\int_{V_L} \rho \underline{u} dV_L\right\} \tag{Bea2}$$

The problem in developing this relation further is that the right hand side requires us to evaluate the rate of change with time of an integral whose limits (V_L) are changing with time and whose integrand is changing with time. The key step in addressing these difficulties is to use the transport theorem to write the right hand side as

$$\underline{F} = \int_V \frac{\partial(\rho\underline{u})}{\partial t} dV + \int_S (\rho\underline{u})\underline{u}.d\underline{S} = \int_V \frac{\partial(\rho\underline{u})}{\partial t} dV + \int_S \rho\underline{u}(\underline{u}.\underline{n}) dS \tag{Bea3}$$

so that the integrals are all transformed to volume or surface integrals over the fixed Eulerian volume. [Note that the left hand side remains the same since the force acting on the coincident Lagrangian volume is the same as the force acting on the Eulerian volume.] This is the linear momentum equation or theorem.

In words it states that for any arbitrary Eulerian volume the vector force acting on that volume is equal to the rate of change of the momentum contained within the Eulerian volume (the first term on the right hand side) plus the net vector flux of momentum **out of** that Eulerian volume. For ease of evaluation note that the momentum flux term is the integral over the surface S of the mass flow rate out of the surface ($\rho u_n dS$ where u_n is the outward velocity component normal to the surface) times the vector velocity, $\underline{u}$.

In other words the linear momentum theorem says that the vector force does two things: it can accelerate the fluid within the Eulerian control volume and it can produce a net flux of momentum **out of** the control volume. This is an important and powerful result which does not involve any assumption about the fluid (whether compressible or incompressible, viscous or inviscid) or, indeed, whether the content of the control volume is solid, fluid or both. In steady flows the first component of the right hand side is zero since the partial time derivatives are zero. Consequently, in steady flow, the net force is equal to the net flux of momentum out of the control volume.

It is important to stress that the momentum equation (Bea3) is a vector equation. However, it is sometimes easier to apply the theorem in its vectorial components

$$F_i \;=\; \int_V \frac{\partial(\rho u_i)}{\partial t} dV \;=\; \int_V \frac{\partial(\rho u_i)}{\partial t} dV \;+\; \int_S u_i(\rho u_n dS) \tag{Bea4}$$

where u_n is the component of the vector velocity normal to dS and out of the control volume. In words this states that the force on the control volume in the i direction is equal to the rate of change of the momentum in the i direction (called the "i-momentum" for convenience) contained within the Eulerian volume plus the net flux of i-momentum **out of** that Eulerian volume. It is often easiest to separately apply the linear momentum theorem in each of three Cartesian directions. Note that then the flux of x-momentum (for example) out of a segment of control volume surface would be given by the mass flow rate out of through that segment times the velocity in the x-direction.

To demonstrate the relationship between the momentum equation (Bea3) and the differential versions of Newton's law, namely the Euler equations and the Navier-Stokes equations, it is necessary to develop and substitute expressions for the force, $\underline{F}$. This is most simply done for an inviscid fluid and should result in Euler's equations. In an inviscid fluid the force $\underline{F}$ will consist of body force acting on the volume of fluid in the control volume and forces due to the pressure acting on the surface. The force due to gravity is then
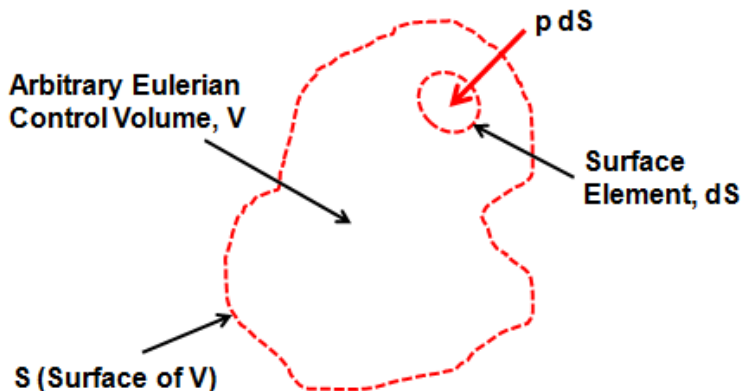

Figure 2: The same arbitrary Eulerian control volume, V.

simply

$$-\int_V \rho g \underline{y}\, dV \tag{Bea5}$$

where g is the acceleration due to gravity and $\underline{y}$ is a unit vector in the vertically upward direction. The force due to the pressure, p, acting on the surface of the control volume is pdS and acts opposite to the outward normal, $\underline{n}$, and so yields the following contribution to $\underline{F}$:

$$- \int_S p\underline{n}\ dS \tag{Bea6}$$

When these two forces are substituted into equation (Bea3), the linear momentum theorem for the inviscid case becomes

$$- \int_V \rho g\underline{y}dV \ - \ \int_S p\underline{n}dS \ = \ \int_V \frac{\partial(\rho\underline{u})}{\partial t}dV \ + \ \int_S \rho\underline{u}(\underline{u}.\underline{n})dS \tag{Bea7}$$

To proceed we employ Gauss' theorem to convert the surface integrals to volume integrals and thus obtain:

$$- \int_V \rho g\underline{y}dV \ - \ \int_V \nabla p\ dV \ = \ \int_V \frac{\partial(\rho\underline{u})}{\partial t}dV \ + \ \int_V \rho(\underline{u}\nabla.\underline{u} + (\underline{u}.\nabla)\underline{u})\ dV \tag{Bea8}$$

where the last term is most readily obtained by considering Gauss' theorem applied to each of the components of $\underline{u}(\underline{u}.\underline{n})$. By continuity, $\nabla.\underline{u} = 0$, and after rearranging equation (Bea8) becomes

$$\int_V \left\{ \rho\frac{\partial\underline{u}}{\partial t} + \rho(\underline{u}.\nabla)\underline{u} + \nabla p + \rho g\underline{y} \right\} dV \ = \ 0 \tag{Bea9}$$

and since the volume V is arbitrary the integrand must be zero everywhere so that

$$\frac{\partial\underline{u}}{\partial t} + (\underline{u}.\nabla)\underline{u} + \frac{\nabla p}{\rho} + g\underline{y} \ = \ 0 \tag{Bea10}$$

which is precisely Euler's equation for an incompressible, inviscid flow.