RAYLEIGH AND EKMAN FLOWS

Some valuable exact solutions to the Navier-Stokes equations involve planar flows in a viscous, incompressible fluid (with constant and uniform viscosity) bounded by a single flat plate that moves in its own plane as depicted in Figure 1. Two classic solutions will be presented here. In the first the fluid is initially at

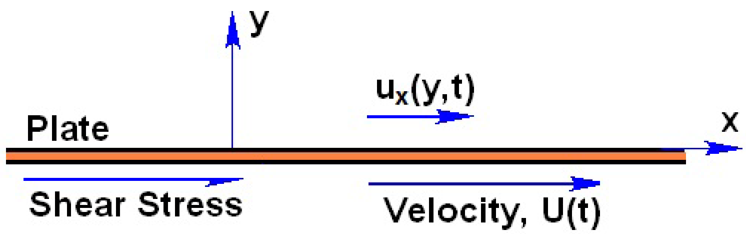

Figure 1: Rayleigh and Ekman flows.

rest and the plate begins to move with a constant velocity, U, in its own plane at time, $t = 0$. This is known as Rayleigh flow. In the second the plate is oscillating with velocity, $U = U^* \sin \hat{f}t$, in its own plane with radian frequency, $\hat{f}$. This is known as Ekman flow. Indeed other solutions of this type are viable, for example, $U = U^* e^{\alpha t}$. We begin by delineating the equations that apply to this whole class of flows.

Since the flow at every x location is the same, it must be true that $u_y = u_z = 0$ and $\partial p/\partial x = 0$ and therefore these uni-directional flows must satisfy the following Navier-Stokes equation for $u_x(y,t)$:

$$\frac{\partial u_x}{\partial t} = \nu \frac{\partial^2 u_x}{\partial x^2} \tag{Bif1}$$

which is a diffusion equation featuring a diffusivity of ν. Since the vorticity, ω, in these flows is

$$\omega = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} = -\frac{\partial u_x}{\partial y} \tag{Bif2}$$

it follows that the vorticity satisfies a similar diffusion equation:

$$\frac{\partial u_x}{\partial t} = \nu \frac{\partial^2 u_x}{\partial x^2} \tag{Bif3}$$

Indeed this illustrates the typical process of diffusion that vorticity satisfies. We now examine the particular solutions, starting with Ekman flow and proceeding to Rayleigh flow.

Ekman flow is solved by deploying separation of variables to write the solution to equation (Bif1) in the form

$$u_x = F(y)G(t) \tag{Bif4}$$

where, after substitution into equation (Bif1), the initially unknown functions, $F(y)$ and $G(t)$, are found to satisfy

$$\frac{1}{G}\frac{dG}{dt} = -k^2 = -\frac{\nu}{F}\frac{d^2F}{dy^2} \tag{Bif5}$$

where k is an arbitrary constant. Solving these two implied ordinary differential equations it follows that

$$u_x = \{C_1 \sin(kt) + C_2 \cos(kt)\}\left\{C_3\; e^{(k/\nu^{1/2})y} + C_4 e^{-(k/\nu^{1/2})y}\right\} \tag{Bif6}$$

where C_1, C_2, C_3, and C_4 are arbitrary constants to be determined. We now apply the boundary conditions. First since the velocity must tend to zero as $y \to \infty$ it follows that $C_3 = 0$ and we can choose $C_4 = 1$ without loss of generality. Second the no-slip condition at the plate, namely

$$(u_x)_{y=0} \;=\; U^* \sin(\hat{f}t) \tag{Bif7}$$

requires that

$$C_2 \;=\; 0 \qquad \text{and} \qquad C_1 \;=\; U^* \qquad \text{and} \qquad k \;=\; \hat{f} \tag{Bif8}$$

so the solution becomes

$$u_x \;=\; U^* \sin(kt)\; e^{-(\hat{f}/\nu^{1/2})y} \tag{Bif9}$$

and the corresponding vorticity is

$$\omega \;=\; -\frac{\hat{f}U^*}{\nu^{1/2}} \sin(kt)\; e^{-(\hat{f}/\nu^{1/2})y} \tag{Bif10}$$

A typical velocity profile for Ekman flow is shown in Figure 2. The velocity oscillates like a standing wave

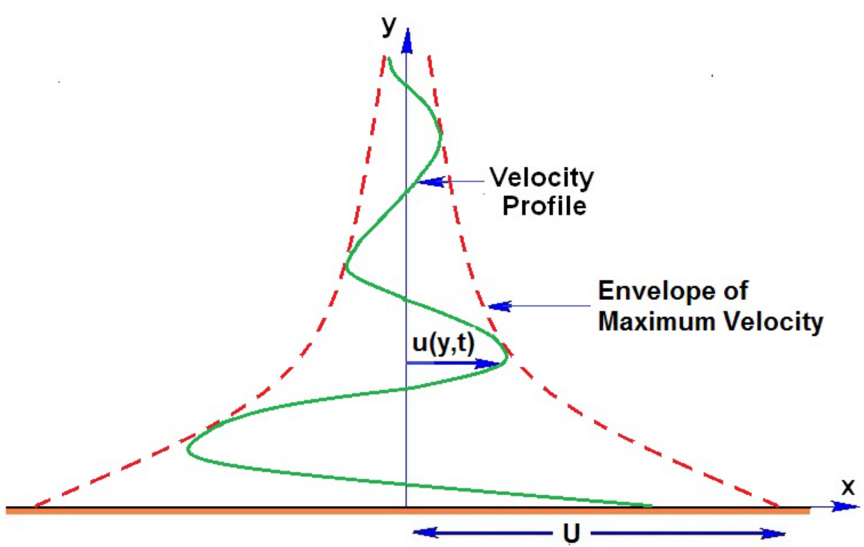

Figure 2: Profiles of velocity and maximum velocity for Ekman flow.

with an amplitude that declines exponentially with distance from the plate. The vorticity profile is similar in kind. The decrease in the amplitudes with y allows definition of a boundary layer thickness, δ, as the distance from the plate at which the velocity magnitude (or the vorticity magnitude) has decreased to 1% of the value at the plate surface. Since the value of $e^{-s} = 0.01$ when $s = 4.605$ it follows that

$$\delta \;=\; 4.605 \frac{\nu^{1/2}}{\hat{f}} \tag{Bif11}$$

As one might have expected the boundary layer thickness decreases as the frequency increases but increases as the kinematic viscosity increases.

We turn now to Rayleigh flow in which the plate is suddenly set in motion with velocity U at time $t = 0$. As in other, equivalent unsteady diffusion problems (for example, in heat transfer) it proves to be appropriate to seek a similarity solution in the case of a sudden change in the boundary condition. The appropriate similarity variable is s where $s = y/(4\nu t)^{1/2}$ (the 4 and the ν are not necessary but are included for later

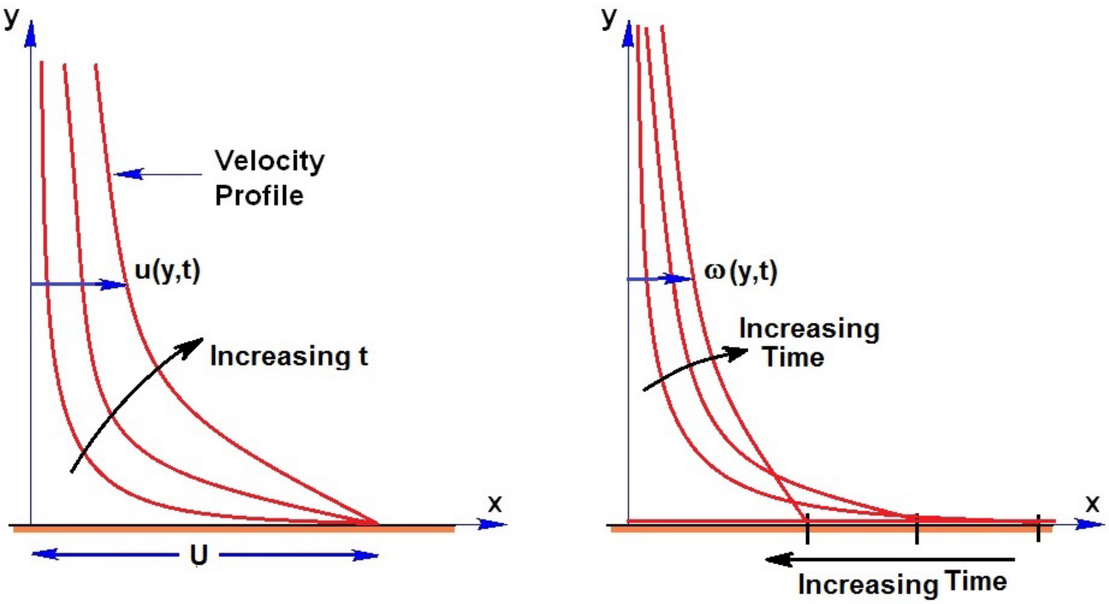

Figure 3: Profiles of velocity (left) and vorticity (right) for Rayleigh flow.

convenience) and, with this similarity variable, the partial differential equation (Bif1) can be reduced to the ordinary differential equation

$$\frac{d^2u_x}{ds^2} + 2s\frac{du_x}{ds} \;=\; 0 \tag{Bif12}$$

where the advantage of including the 4 and the ν in the definition of the similarity variable becomes apparent because it follows that this governing equation (Bif12) contains no parameters. Moreover the boundary conditions the solution must satisfy are

- $u_x \to 0$ as $y \to \infty$ that is as $s \to \infty$.
- $u_x = 0$ for $t = 0$, $y > 0$, that is at $s = \infty$.
- $u_x = U$ for $y = 0$, $t > 0$, that is at $s = 0$.

The solution to equation (Bif12) is obtained by setting

$$q \;=\; \frac{du_x}{ds} \tag{Bif13}$$

so that equation (Bif12) becomes

$$\frac{dq}{q} \;=\; -2s\;ds \qquad \text{and so} \qquad q \;=\; C_1\;e^{-s^2} \tag{Bif14}$$

which can then be integrated to yield

$$u_x \;=\; C_1\int_0^s e^{-z^2}\;dz\; + C_2 \tag{Bif15}$$

where C_1 and C_2 are integration constants and z is a dummy s variable. Substituting back from the definition of s this solution can be written as

$$u_x \;=\; C_1\int_0^{y/(4\nu t)^{1/2}} e^{-z^2}\;dz\; + C_2 \;=\; \frac{C_1(\pi)^{1/2}}{2}erf\left(\frac{y}{(4\nu t)^{1/2}}\right) + C_2 \tag{Bif16}$$

where $erf(\ )$ is the error function. Then applying the three boundary conditions listed above it transpires that $C_2 = U$ and $C_1 = -2U/(\pi)^{1/2}$ so that the final solution is

$$u_x \ = \ U\left[1 - erf\left(\frac{y}{(4\nu t)^{1/2}}\right)\right] \qquad \text{(Bif17)}$$

and

$$\omega \ = \ \frac{U}{(4\nu t)^{1/2}}\ e^{-\frac{y^2}{4\nu t}} \qquad \text{(Bif18)}$$

The form of this solution is shown graphically in Figure 3. The velocity profile expands outward as time progresses. It is instructive to quantify the thickness of the boundary layer (δ) that has been affected by the motion of the plate by seeking the distance from the plate at which the velocity has decreased to $0.01U$. Since $erf(1.82) = 0.99$ the thickness $\delta = 3.64(\nu t)^{1/2}$. Thus the boundary layer thickness increases with the square root of time and also increases as the viscosity increases. The vorticity begins at $t = 0$ as an infinitely line line of infinite vorticity. That vorticity then diffuses out into the fluid as the value at the surface of the plate decreases. Notice also how in the absence of viscosity ($\nu = 0$) the velocity must be zero throughout the fluid; in other words the no-slip condition cannot be satisfied.

These features of Rayleigh flow are particularly instructive. They reveal how the no-slip condition produces vorticity which then diffuses out into the fluid through the action of viscosity. This is a common feature in all flows involving solid surfaces.