POISEUILLE FLOW

Poiseuille flow is the steady, axisymmetric flow in an infinitely long, circular pipe of radius, R, as sketched in Figure 1. The flow is caused by a pressure gradient, dp/dx, in the axial direction, x. The resulting

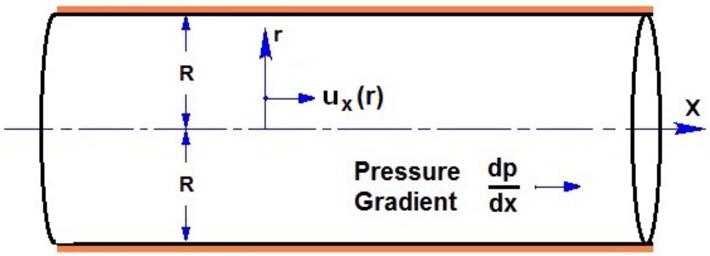

Figure 1: Poiseuille flow.

axisymmetric continuity equation for an incompressible fluid yields

$$\frac{\partial u_x}{\partial x} = 0 \tag{Bic1}$$

so that the axial velocity, $u_x(r)$, is a function only of r, the radial coordinate. Using this the axisymmetric Navier-Stokes equations for an incompressible fluid of constant and uniform viscosity reduce to

$$\frac{\partial p}{\partial x} = \mu \left\{ \frac{\partial^2 u_x}{\partial r^2} + \frac{1}{r}\frac{\partial u_x}{\partial r} \right\} \tag{Bic2}$$

$$\frac{\partial p}{\partial r} = 0 \tag{Bic3}$$

The second of these shows that the pressure, $p(x)$, is a function only of x and hence the gradient, dp/dx, is well defined and a parameter of the problem. This allows the first of these equations (Bic2) to be integrated so that the velocity, u_x, can be written as

$$u_x(r) = \frac{r^2}{4\mu}\left(\frac{dp}{dx}\right) + C_1 \ln r + C_2 \tag{Bic4}$$

where C_1 and C_2 are integration constants to be determined by the application of the boundary conditions. On the axis the velocity cannot be infinite therefore C_1 must be zero. Moreover, the no-slip boundary condition on the pipe wall requires that $u_x(R) = 0$ and so

$$C_2 = -\frac{R^2}{4\mu}\left(\frac{dp}{dx}\right) \tag{Bic5}$$

and so the solution to Poiseuille flow in a circular pipe is

$$u_x(r) = \frac{1}{4\mu}\left(-\frac{dp}{dx}\right)(R^2 - r^2) \tag{Bic6}$$

where the pressures, p_1 and p_2, could be measured at two different x locations a distance ℓ apart in order to determine $dp/dx = (p_1 - p_2)/\ell$. Parenthetically, we should add that the general solution (Bic4) also allows construction of the axial flow between two cylinders in which the outer radius of the inner cylinder is R_1 and the inner radius of the outer cylinder is R_2. The no-slip boundary conditions at these two surfaces then require that

$$\frac{R_1^2}{4\mu}\left(\frac{dp}{dx}\right) + C_1 \ln R_1 + C_2 \;=\; 0 \tag{Bic7}$$

and

$$\frac{R_2^2}{4\mu}\left(\frac{dp}{dx}\right) + C_1 \ln R_2 + C_2 \;=\; 0 \tag{Bic8}$$

from which C_1 and C_2 can be determined and the solution constructed. Beyond that it is also possible to stipulate that one of the cylinders has an axial velocity, U, and to proceed to construct yet another axisymmetric flow of this type.

However, we confine the present discussion to the simple Poiseuille flow while recognizing that parallel analyses can be carried out for the other variants. Note first that the velocity distribution in Poiseuille flow is parabolic according to equation (Bic6) and has a maximum velocity on the axis of

$$u_x(0) \;=\; \frac{R^2}{4\mu}\left(-\frac{dp}{dx}\right) \tag{Bic9}$$

The volume flow rate, $\dot{Q}$, is

$$\dot{Q} \;=\; \int_0^R 2\pi r \; u_x \; dr \;=\; \frac{\pi R^4}{8\mu}\left(-\frac{dp}{dx}\right) \tag{Bic10}$$

so that the average velocity of the flow, $\overline{u}$, is

$$\overline{u} \;=\; \frac{\dot{Q}}{\pi R^2} \;=\; \frac{R^2}{8\mu}\left(-\frac{dp}{dx}\right) \tag{Bic11}$$

Thus the average is 1/2 of the maximum. This last equation is often written as

$$\Delta p \;=\; \frac{8\mu\ell\overline{u}}{R^2} \tag{Bic12}$$

to yield the pressure drop, Δp, over a length, ℓ, of the pipe.

The shear stress distribution in the flow is best examined by applying the momentum theorem to a cylindrical control volume of radius, r, centered on the axis of the pipe and with length, ℓ . Since the velocities in an out of the end of this cylinder are identical there is no net flux of momentum in or out of this control volume and so the theorem says the axial forces must balance. The forces due to the pressure on one end of this cylinder will be $\pi r^2 p$ while the force on the other end a distance ℓ downstream will be $\pi r^2(p-\Delta p)$ where Δp is the pressure drop between the two locations. Consequently the net axial force in the positive x direction due to the pressure forces on the ends will be $\pi r^2 \Delta p$. The only other force acting in the axial direction is due to the shear stress acting on the outer surface area of the control volume. Denoting that shear stress by σ_{rr} which, in accord with the sign convention used in section (Bhd), is positive in the positive x direction, this shear force will be equal to $2\pi r\ell\sigma_{rr}$. Therefore the balance of forces acting on the control volume requires that

$$2\pi r\ell\sigma_{rr} + \pi r^2 \Delta p \;=\; 0 \tag{Bic13}$$

so that

$$\sigma_{rr} = -\frac{r\Delta p}{2\ell} \tag{Bic14}$$

a result which is completely independent of the constitutive law for the contents of the pipe. The negative sign is also in accord with expectations since the surrounding fluid necessarily counteracts the positive force due to the pressures on the ends of the control volume. Furthermore we note that the shear stress in the pipe always varies linearly with the radial location, r.

Normally the sign convention for the wall shear stress, τ_w, that the fluid applies to the interior surface of the pipe is such that

$$\tau_w \;=\; -(\sigma_{rr})_{r=R} \;=\; \frac{R\Delta p}{2\ell} \tag{Bic15}$$

which for Poiseuille flow, using the relation (Bic10) leads to

$$\tau_w \;=\; \frac{4\mu\overline{u}}{R} \tag{Bic16}$$

This is a convenient point introduce some conventional engineering definitions. In the internal flow through any component, the coefficient of loss is denoted by k and defined as $k = 2\Delta p/\rho\overline{u}^2$. In a pipe flow a further coefficient, the friction factor, f, is introduced to take account of the fact that the loss k will be proportional to the length, ℓ, of the pipe. Specifically, the friction factor, f, is defined as

$$f \;=\; \frac{2Rk}{\ell} \;=\; \frac{4R\Delta p}{\rho\ell\overline{u}^2} \tag{Bic17}$$

Inserting the result (Bic16) for Poiseuille flow this yields

$$f \;=\; \frac{4R\Delta p}{\rho\ell\overline{u}^2} \;=\; \frac{32\mu}{\rho R\overline{u}} \;=\; \frac{64}{Re} \tag{Bic18}$$

where $Re = 2\rho\overline{u}R/\mu$ is the Reynolds number for the pipe flow. This classic result is, of course, only valid for laminar flow; when the flow transitions to turbulent other factors enter the picture but this discussion is best delayed until later sections. In practical terms that transition typically occurs when Re reaches a value of about 2000 though many other factors may effect that value.

The expression (Bic18) is only one feature that appears in perhaps the most widely used chart in all of engineering fluid mechanics, the graph of f against the Reynolds number that is known as the Moody chart. It can be seen on the left hand side of Figure 2.

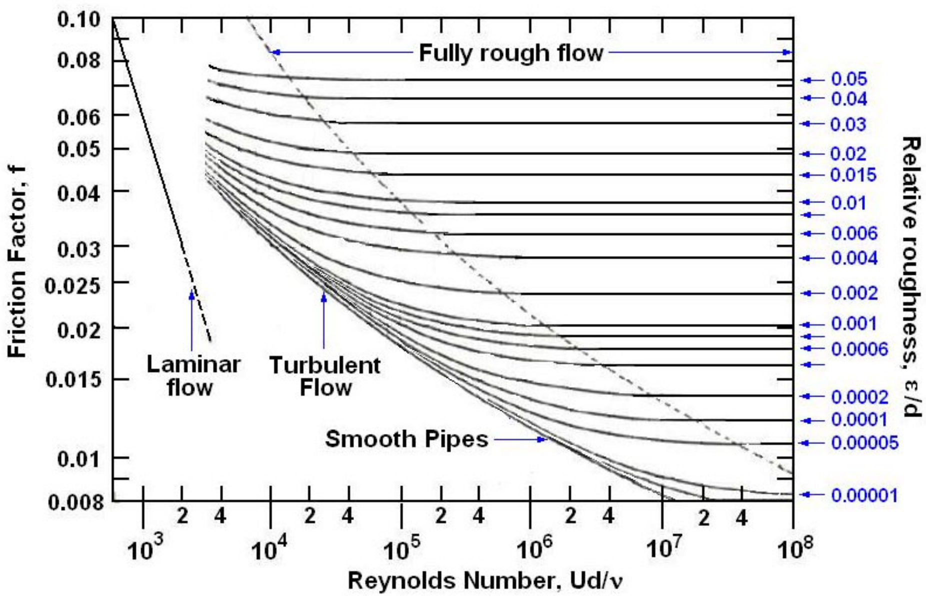

Figure 2: The Moody diagram for the friction factor, f, as a function of Reynolds number, $2\rho\overline{u}R/\mu$.