LAMINAR ROUND JET

Remarkable among the exact solutions to the Navier-Stokes equations for incompressible flow with uniform and constant density is the solution for a *laminar round jet* discovered by Slezkin (1934), Landau (1944), and Squire (1951). The uniqueness lies in the fact that it is the only known solution which is not unidirectional (though Yih (1969) suggests it may be just one of a family of similar solutions). It is special in part because, unlike solutions for potential flow or Stokes flow, these basic solutions for the Navier-Stokes equations are not superposable and therefore more complicated solutions cannot be generated by combining the basic solutions. For details of the solution for the laminar round jet the reader is referred to Squire (1951) or Yih (1969); only an outline will be given here. We note that the solution can be interpreted as that due to a point source of momentum of strength, $\mathcal{M}$, or point force. We note that in the limit of zero Reynolds number where the Navier-Stokes equations become the equations for Stokes flow, this solution for a round laminar jet in an inertialess fluid becomes the fundamental singularity known as a *stokeslet* (see section (Blc)). However, unlike those Stokes flows, the nonlinearity of the Navier-Stokes equations prevents superposition of laminar round jet solutions that could form more complicated flows.

The axisymmetric flow known as the laminar round jet is constructed in a spherical coordinate system, (r, θ, ϕ), with the point source at the origin, $\theta = 0$ being the direction of the momentum source (or singular force). The flow is independent of ϕ. The Stokes streamfunction is assumed to be of the form

$$\psi(r, \theta, \phi) \;=\; \nu r F(\theta) \tag{Big1}$$

where $F(\theta)$ is to be determined. Then the velocities become

$$u_r \;=\; \frac{\nu}{r\sin\theta}\frac{dF(\theta)}{d\theta} \qquad \text{and} \qquad u_\theta \;=\; -\frac{\nu}{r\sin\theta}F(\theta) \tag{Big2}$$

With these relations, the Navier-Stokes equations in spherical coordinates become

$$-\frac{1}{r}(u_r^2 + u_\theta^2) + \frac{\nu}{r}\frac{\partial u_r}{\partial\theta} \;=\; -\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{\nu}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_r}{\partial\theta}\right) \tag{Big3}$$

$$\frac{\nu}{r}\frac{\partial u_\theta}{\partial\theta} \;=\; -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} + \frac{\nu}{r^2}\frac{\partial u_r}{\partial\theta} \tag{Big4}$$

with $u_\phi = 0$. Squire's solution, which satisfies these equations is

$$F(\theta) \;=\; \frac{2\sin^2\theta}{a + 1 - \cos\theta} \tag{Big5}$$

where a is an arbitrary parameter of the solution. It transpires that the magnitude of the point source of momentum, $\mathcal{M}$, (or point force) is given by

$$\frac{\mathcal{M}}{2\pi\rho\nu^2} \;=\; \frac{32(a+1)}{3a(a+2)} + 8(a+1) - 4(a+1)^2\ln(1 + 2/a) \tag{Big6}$$

Streamlines for three choices of a (or $\mathcal{M}$) are shown in Figure 1, namely $a = 1.0$ ($\mathcal{M}/\rho\nu^2 = 34.76$), $a = 0.1$ ($\mathcal{M}/\rho\nu^2 = 314.0$) and $a = 0.01$ ($\mathcal{M}/\rho\nu^2 = 3282$). The acceleration of the fluid as it passes from left to right is clear in these graphs. The increase in the acceleration with increasing $\mathcal{M}$ is also evident.

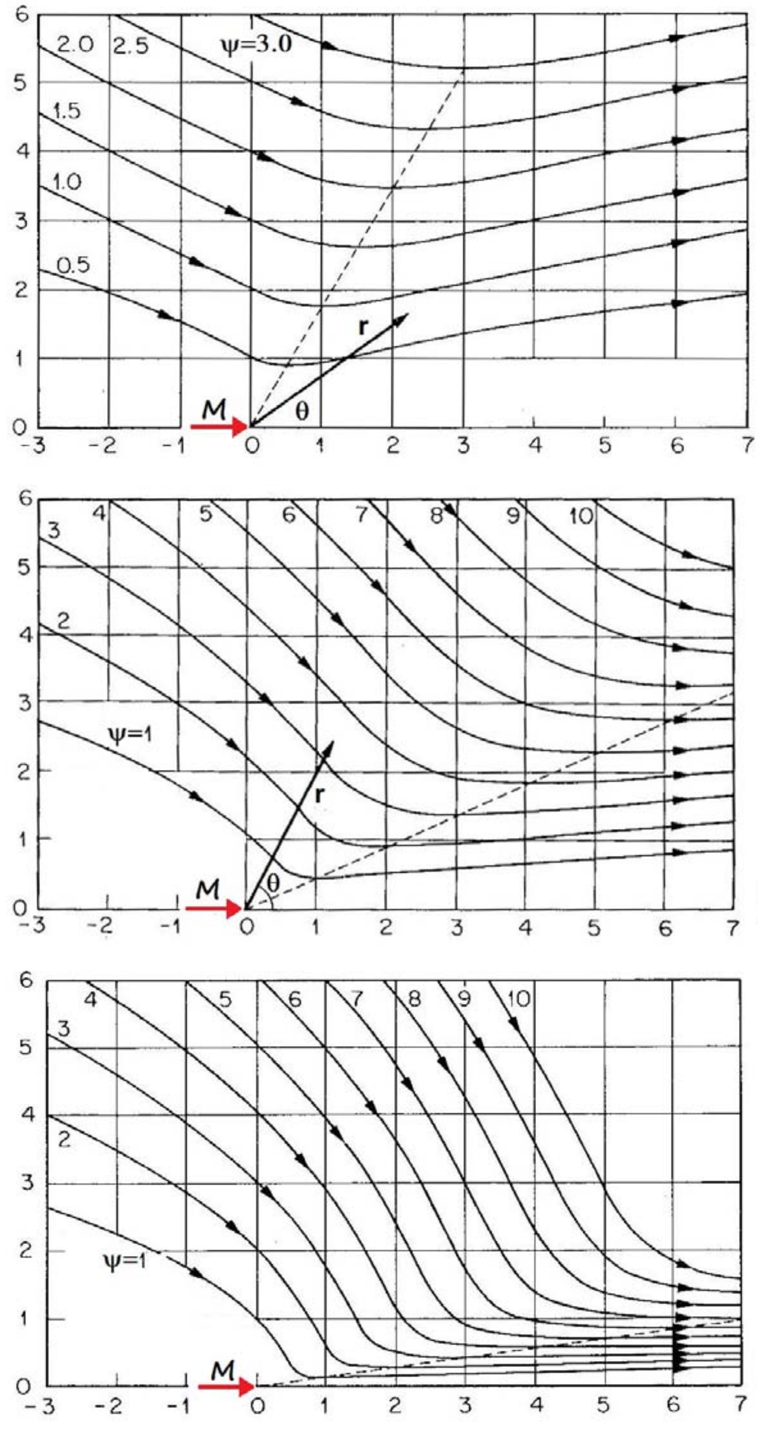

Figure 1: Laminar round jets for $a = 1$ (top), $a = 0.1$ (middle) and $a = 0.01$ (bottom). The horizontal coordinate is $r\cos\theta$, the vertical coordinate is $r\sin\theta$, and the value on each streamline is ψ. Adapted from Squire(1951).