VORTEX FLOW

Some useful exact solutions to the Navier-Stokes equations also emerge from flows in cylindrical coordinates in which the only non-zero component of velocity is the circumferential velocity, u_θ. These are vortex flows which, as depicted in Figure 1 could either have an exterior cylindrical boundary, an interior cylindrical boundary or both. In planar flow and polar coordinates, the continuity equation (Bce7) simply yields

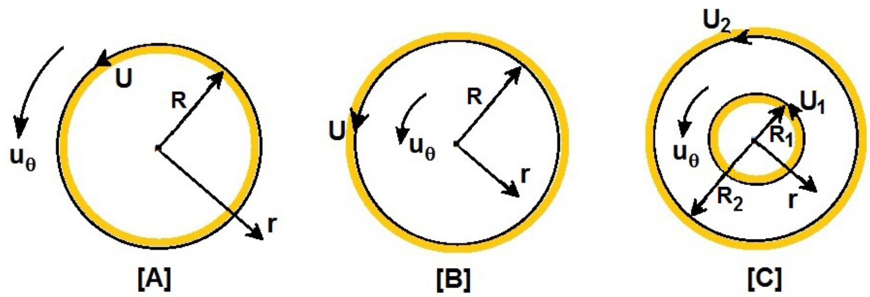

Figure 1: Three types of vortical flow.

$\partial u_\theta / \partial \theta = 0$ which could be regarded as obvious since otherwise u_θ would be double-valued. The Navier-Stokes equations in section (Bhg) with $u_r = u_z = 0$, derivatives in the z direction set equal to zero and derivatives in the θ direction set equal to zero since otherwise quantities would be double-valued, simply reduce to two consequential equations

$$\rho \frac{u_\theta^2}{r} = \frac{\partial p}{\partial r} \tag{Bie1}$$

and

$$\frac{\partial u_\theta}{\partial t} = \nu \left\{ \frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} \right\} \tag{Bie2}$$

We will focus on the steady flows that emerge from these equations though there are unsteady flows that involve the viscosity. The second equation (Bie2) can be simply solved to yield

$$u_\theta = C_1 r + \frac{C_2}{r} \tag{Bie3}$$

where C_1 and C_2 are constants to be determined by the boundary conditions. The equation (Bie1) can be integrated to yield

$$\frac{p}{\rho} = \frac{C_1^2 r^2}{2} + 2C_1 C_2 \ln r - \frac{C_2^2}{2r^2} + C_3 \tag{Bie4}$$

where C_3 is another integration constant. Now we apply the boundary conditions to each of the three types of flow sketched in Figure 1:

[A] Type [A] flow requires $u_\theta = U$ on $r = R$ and non-infinite velocity as $r \to \infty$ therefore

$$C_1 = 0 \quad \text{and} \quad C_2 = RU \tag{Bie5}$$

This is simply a potential or "free" vortex. The pressure distribution associated with it is

$$\frac{(p - p_\infty)}{\rho} = -\frac{R^2 U^2}{2r^2} \tag{Bie6}$$

where p_∞ is the pressure at infinity.

[B] Type [B] flow requires $u_\theta = U$ on $r = R$ and non-infinite velocity as $r \to 0$ therefore

$$C_1 = \frac{U}{R} \quad \text{and} \quad C_2 = 0 \tag{Bie7}$$

This is simply solid body rotation or a "forced" vortex. The pressure distribution associated with it is

$$\frac{(p - p_0)}{\rho} = \frac{U^2 r^2}{2R^2} \tag{Bie8}$$

where p_0 is the pressure at the center.

[C] Type [C] flow requires $u_\theta = U_1$ on $r = R_1$ and $u_\theta = U_2$ on $r = R_2$ therefore

$$C_1 = \frac{(R_2 U_2 - R_1 U_1)}{(R_2^2 - R_1^2)} \quad \text{and} \quad C_2 = \frac{R_1 R_2 (R_2 U_1 - R_1 U_2)}{(R_2^2 - R_1^2)} \tag{Bie9}$$

This is mix of free and forced vortex in the gap between the cylinders.

None of these steady, exact solutions involve the viscosity.