# Bernoulli's Equation

Another key feature of irrotational flow is the fact that it allows integration of Euler's equation to yield Bernoulli's equation, one of the most widely used relations in practical fluid flow problems. Euler's equation for inviscid flow with conservative body forces, equation (Bdb6), is

$$\rho\left\{\frac{\partial \underline{u}}{\partial t}+\nabla\left(\frac{|\underline{u}|^2}{2}\right)-\underline{u}\times(\nabla\times\underline{u})\right\} \;=\; -\nabla p+\nabla\mathcal{U} \tag{Bdg1}$$

and, since $\nabla\times\underline{u}=\underline{\omega}$ this can be written as

$$\rho\left\{\frac{\partial \underline{u}}{\partial t}+\nabla\left(\frac{|\underline{u}|^2}{2}\right)-\underline{u}\times\underline{\omega}\right\} \;=\; -\nabla p+\nabla\mathcal{U} \tag{Bdg2}$$

We consider a number of different circumstances in which important scalar relations can be derived from this:

- If the flow is steady ($\partial\underline{u}/\partial t=0$) and irrotational ($\underline{\omega}=0$) as well as inviscid then equation (Fg2) becomes

$$\rho\nabla\left(\frac{|\underline{u}|^2}{2}\right)+\nabla p-\nabla\mathcal{U} \;=\; 0 \tag{Bdg3}$$

- If, in addition, the flow is incompressible so that $\rho$ is uniform and constant then equation (Bdg3) can be written as

$$\nabla\left(\frac{\rho|\underline{u}|^2}{2}+p-\mathcal{U}\right) \;=\; 0 \tag{Bdg4}$$

  Since the gradient of the quantity in parentheses is everywhere zero it follows that that quantity can only be a function of time and since the flow is assumed to be steady it can only be a constant:

$$\frac{\rho|\underline{u}|^2}{2}+p-\mathcal{U}=\text{Constant and Uniform} \tag{Bdg5}$$

  If the only body forces are those due to gravity acting in the $y$ direction so that $\mathcal{U}=-\rho g y$ this becomes

$$\frac{\rho|\underline{u}|^2}{2}+p+\rho g y=\text{Constant} \tag{Bdg6}$$

  This is Bernoulli's equation for steady, incompressible, inviscid, irrotational flow with conservative body forces due to gravity. It is one of the most widely used equations of fluids engineering and we have much more to say about it elsewhere in this text.

- A version of Bernoulli's equation for unsteady flow can be obtained by noting that since the flow is irrotational we can write $\underline{u}=\nabla\phi$ and thus retain the $\partial\underline{u}/\partial t$ term as $\nabla(\partial\phi/\partial t)$ so that Bernoulli's equation for unsteady, incompressible, inviscid, irrotational flow with conservative body forces due to gravity is:

$$\frac{\partial\phi}{\partial t}+\frac{\rho|\underline{u}|^2}{2}+p+\rho g y=\text{Uniform in space but may be function of time} \tag{Bdg7}$$

- What if the flow is not irrotational and we must retain the term $\underline{u}\times\underline{\omega}$ in equation (Bdg2)? Note that this vector quantity $\underline{u}\times\underline{\omega}$ is perpendicular to both the velocity vector $\underline{u}$ and to the vorticity vector $\underline{\omega}$. Consequently the components of $\underline{u}\times\underline{\omega}$ in the direction of both the velocity vector and the vorticity vector are zero. It follows that in any steady, inviscid, incompressible, flow (with a body force potential $\mathcal{U}$) the quantity

$$\frac{\rho|\underline{u}|^2}{2} + p - \mathcal{U} \tag{Bdg8}$$

is constant *along any streamline or along any vortex line.* It is therefore constant on any sheet defined by a set of vortex lines and streamlines. It may however change from one sheet to another if the flow is not irrotational.

Since Bernoulli's equation is used extensively in engineering, some groups of terms in it have traditional names that are useful to mention. Equation (Bdg6) is often written as

$$\frac{|\underline{u}|^2}{2g} + \frac{p}{\rho g} + y = \text{Constant} \tag{Bdg9}$$

The group, $y + p/\rho g$, is referred to as the **static head** or **piezometric head** whereas the term $|\underline{u}|^2/2g$ is called the **dynamic head**. The sum of the static and dynamic heads is called the **total head** (denoted by $H$) and equivalently the quantity $\rho|\underline{u}|^2/2 + p + \rho g y$ is known as the **total pressure** (denoted by $p^T$). Thus Bernoulli's equation (Bdg9) states that if the flow is steady, inviscid, incompressible and irrotational the total head and the total pressure will be the same everywhere in the flow. On the other hand if there are viscous losses the total head (or total pressure) will (presumably) decrease in the direction of flow as energy is diverted to overcome the viscous losses. Indeed, Bernoulli's equation can be viewed as an energy relation and we explore this further in another page.

Bernoulli's equation (Bdg9) is often used in the non-dimensional form

$$\frac{|\underline{u}|^2}{U^2} + \frac{p}{\frac{1}{2}\rho U^2} + \frac{2gy}{U^2} = \text{Constant} \tag{Bdg10}$$

where $U$ is some reference fluid velocity, often that of an upstream uniform flow. Frequently, it is convenient in this context to define a non-dimensional pressure, or *"coefficient of pressure"*, $C_p$, as

$$C_p = \frac{(p - p_\infty)}{\frac{1}{2}\rho U^2} \tag{Bdg11}$$

where $p_\infty$ is a reference pressure often that where the fluid velocity, $U$, pertains and where the elevation is $y_\infty$. Then Bernoulli's equation (Bdg10) can be written as

$$\frac{|\underline{u}|^2}{U^2} + C_p + \frac{2g(y - y_\infty)}{U^2} = \text{Constant} \tag{Bdg12}$$

In many high speed flows the third term on the left-hand side is negligible. Then provided $p_\infty$ and $U$ are chosen to refer to upstream uniform flow conditions then Bernoulli's equation implies that $C_p$ and the velocity ratio, $|\underline{u}|/U$ are simply related by

$$C_p = \left[1 - \frac{|\underline{u}|^2}{U^2}\right]^{\frac{1}{2}} \tag{Bdg13}$$