Energy Implications of Bernoulli's Equation

Bernoulli's equation has important energy implications which are useful to discuss in detail in order to better understand the mechanics of fluid flow. We begin with the first law of thermodynamics which states that the heat added to a given Lagrangian volume plus the work done on that volume is equal to the increase in the stored energy within that volume. Dividing through by the mass of fluid in the Lagrangian volume this law can be written as

$$\Delta q \; + \; \Delta w \; = \; \Delta \left\{ h + \frac{|\underline{u}|^2}{2} + gy \right\} \tag{Fh1}$$

where Δq is the heat added per unit mass, Δw is the work done on the Lagrangian volume per unit mass and Δh is the increase in the enthalpy stored in the Lagrangian volume per unit mass. By definition $h = e + p/\rho$ where e is the internal energy per unit mass. Unlike the Lagrangian volumes that are normally treated in thermodynamics, this volume is moving and perhaps accelerating and so, in addition to the stored enthalpy, energy is also stored in the Lagrangian volume as kinetic energy and as potential energy. Consequently the terms $\Delta(|\underline{u}|^2/2)$ and Δgy are the increases in the kinetic energy and potential energy per unit mass stored in the Lagrangian volume. We should note that the term Δw does not include the work done by surface forces (pressure) or by gravity since these have been incorporated in the terms on the right hand side of equation (Fh1). We also note that, for simplicity, we have omitted terms in the work done by viscous surface stresses. We have also omitted some other, possibly relevant forms of stored energy such as chemical energy (that may change due to chemical reactions) or electromagnetic energy.

More general applications of the first law are described in other pages. The purpose here is to demonstrate how this applies in steady, inviscid and irrotational flow of an incompressible fluid. In this class of flows there is no mechanism for kinetic energy to become thermal energy (or vice versa) since there are no viscous effects to absorb kinetic energy and no compressible effects to cause compressive heating. Moreover, there is no mechanisms which would convert the added heat, Δq into kinetic energy. Therefore equation (Fh1) splits into two independent parts namely

$$\Delta q \; = \; \Delta e \quad \text{and} \quad \Delta w \; = \; \Delta \left\{ \frac{p}{\rho} + \frac{|\underline{u}|^2}{2} + gy \right\} \tag{Fh2}$$

This greatly simplifies the solution of heat transfer and fluid flow problems in incompressible, inviscid fluid mechanics. The right hand equation can be used in the solution of the flow problem while the left hand equation separately contributes to the solution of the heat transfer in the flow. As we see elsewhere, these problems are much more closely coupled in compressible flow where the temperature distribution in the flow effects the pressure and density and therefore the velocity field.

Focussing here on the equations governing the flow, we see that the first law applied to incompressible, inviscid flow leads to

$$\Delta w \; = \; \Delta \left\{ \frac{p}{\rho} + \frac{|\underline{u}|^2}{2} + gy \right\} \; = \; \Delta \left\{ gH \right\} \tag{Fh3}$$

where Δw is the external work done on the Lagrangian volume per unit mass excluding the work done by surface forces and by gravity. In many flows $\Delta w = 0$ for no other work is done on the Lagrangian volume. In those circumstances we recover Bernoulli's equation:

$$\left\{ \frac{p}{\rho} + \frac{|\underline{u}|^2}{2} + gy \right\} \; = \; \text{Constant} \tag{Fh4}$$

However, there are also many exceptions in which $\Delta w \neq 0$. There are two common circumstances in which these non-zero values of Δw needs to be identified:

- When the Lagrangian fluid element passes through a device like a pump or turbine with blades which do work on the fluid, then that work needs to be included as a non-zero contribution to Δw. Specifically, if the rate of work done *on the fluid* per unit time by the pump or other active device is denoted by $\dot{W}$ and the mass flow rate of fluid through the device is denoted by m, then the work done on the fluid per unit mass is $\dot{W}/m$ and it follows that the increase in total head, H, across the pump (or other device) is equal to $\dot{W}/(gm)$:

$$H_2 - H_1 \;=\; \left\{ \frac{p}{(\rho g)} + \frac{|\underline{u}|^2}{(2g)} + y \right\}_2 \;-\; \left\{ \frac{p}{(\rho g)} + \frac{|\underline{u}|^2}{(2g)} + y \right\}_1 \;=\; \frac{\dot{W}}{(gm)} \tag{Fh5}$$

  where the subscripts 1 and 2 respectively refer to the inflow to and discharge from the pump or other device. Clearly a device such as a turbine which removes energy from the fluid has a negative value of $\dot{W}$.

- Though Bernoulli's equation applies to inviscid flow, we may, in general terms, incorporate the effects of viscosity by recognizing that this will cause mechanical energy to be dissipated into heat and so have the same effect as a turbine or other machine that removes mechanical energy from the flow. The consequence in the flow of a viscous fluid through a passive component like a duct or valve would be that the inlet (subscript 1) and discharge (subscript 2) total heads would be related by

$$H_1 - H_2 \;=\; \left\{ \frac{p}{(\rho g)} + \frac{|\underline{u}|^2}{(2g)} + y \right\}_1 \;-\; \left\{ \frac{p}{(\rho g)} + \frac{|\underline{u}|^2}{(2g)} + y \right\}_2 \;=\; \Delta H \;>\; 0 \tag{Fh6}$$

  Empirical expressions for the ΔH due to viscous effects in many different components and fittings will be explored and described in a later page.