Kelvin's Theorem

Kelvin's theorem is an outgrowth of the previously described properties of vorticity and circulation. It

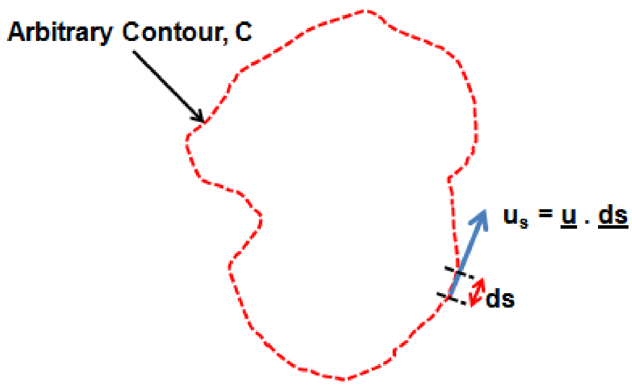

Figure 1: Circulation around an arbitrary closed contour in a flow.

states that the circulation, Γ, around any closed contour, C, in the inviscid flow of a barotropic fluid with conservative body forces does not change with time. Recall that for any closed contour, C (see Figure 1), the circulation is defined as the line integral of the tangential velocity around any closed contour, C, in the flow:

$$\Gamma \;=\; \oint_C u_s ds \;=\; \oint_C \underline{u}.\underline{ds} \tag{Bdj1}$$

where $\underline{u}$ is the fluid velocity vector, $\underline{ds}$ is an vector element along the closed contour and the circle on the integral sign indicates a closed contour. Taking the Lagranian derivative of this yields

$$\frac{D\Gamma}{Dt} \;=\; \oint_C \frac{D\underline{u}}{Dt}.\underline{ds} \;+\; \oint_C \underline{u}.\frac{D\underline{ds}}{Dt} \tag{Bdj2}$$

The governing equation for an inviscid fluid with conservative body forces (see equation (Bdb4)) is

$$\frac{D\underline{u}}{Dt} \;=\; \left\{\frac{\partial \underline{u}}{\partial t} + (\underline{u}\;.\nabla)\underline{u}\right\} \;=\; -\frac{1}{\rho}\nabla p + \nabla \mathcal{U} \tag{Bdj3}$$

where $\mathcal{U}$ is the body force potential.

Using this, the first term on the right hand side of equation (Bjd2) becomes

$$\oint_C \frac{D\underline{u}}{Dt}.\underline{ds} \;=\; \int_A \nabla \times \left(-\frac{1}{\rho}\nabla p + \nabla \mathcal{U}\right).\underline{n}dS \;=\; \int_A \frac{1}{\rho^2}\left(\nabla\rho \times \nabla p\right).\underline{n}dS \;=\; 0 \tag{Bdj4}$$

where Stokes' theorem has been used to convert the line integral to a surface integral over the area A enclosed by the contour C, $\underline{n}$ is a unit normal to that surface and dS is an element of A. When that area cuts through an object the subsequent analysis must be changed but we will postpone discussion of this until later; for now it is assumed that A can be created wholly within the fluid. The end result on the right of equation (Bdj4) follows from the assumption of a barotropic fluid.

The second term on the right hand side of equation (Bjd2) can be developed using the identity $D\underline{ds}/Dt = (\underline{ds}.\nabla)\underline{u}$ and Stokes' theorem to yield:

$$\oint_C \underline{u}.\frac{D\underline{ds}}{Dt} \;=\; \oint_C \underline{u}.\left\{(\underline{ds}.\nabla)\underline{u}\right\} \;=\; \oint_C \nabla\left(|\underline{u}|^2\right).\underline{ds} \;=\; 0 \qquad \text{(Bdj5)}$$

Using both results (Bdj4) and (Bdj5) we have proved that

$$\frac{D\Gamma}{Dt} \;=\; 0 \qquad \text{(Bdj6)}$$

for an inviscid fluid with conservative body forces.