Vorticity

The vorticity, $\underline{\omega}(\underline{x}, t)$ or $\omega_i(x_i, t)$, in a fluid flow is a vector quantity equal to twice the rate of rotation of an infinitesmal fluid element. It is simply related to the velocity vector by

$$\underline{\omega} = \nabla \times \underline{u} \tag{Bdd1}$$

$$\omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \quad ; \quad \omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \quad ; \quad \omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{Bdd2}$$

Note that just as we defined streamlines to be lines that are everywhere tangential to the velocity vector, $\underline{u}$, we also define vortex lines to be lines that are everywhere tangential to the vorticity vector, $\underline{\omega}$. Note also that by virtue of the definition of vorticity, vortex lines are everywhere orthogonal to streamlines. Also note that in planar flow the vorticity is perpendicular to the plane of the flow; in such flow it is convenient to use ω to denote the magnitude of the vorticity and therefore

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{Bdd3}$$

Though it may be premature to do so, it is useful to anticipate some of the properties of vorticity even though they will not be proven until later. We begin with a uniform stream for which $\underline{\omega} = 0$ since all the velocity gradients are zero. It transpires that when such a flow encounters a solid object such as the

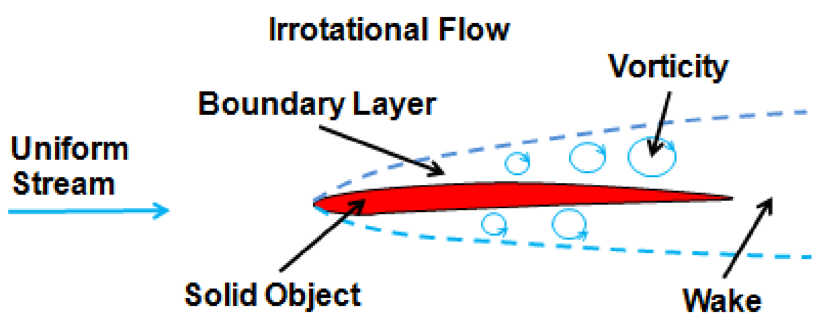

Figure 1: Boundary layer at large Reynolds numbers.

airfoil in figure 1, vorticity is created at the surface of that object since the fluid in contact with the solid surface is at rest as a result of the no-slip condition. Consequently the fluid is inclined to roll at the surface rather like a layer of microscopic ball bearings between two sliding surfaces. But due to the action of the tangential stresses between fluid layers (caused by the fluid viscosity), this rotation tends to be transmitted further outward into the fluid. Hence the vorticity is created at the solid surface due to the no-slip condition and is then diffused outward into the fluid through the action of viscosity. Of course those outer layers of fluid are also being carried along in the flow and therefore the vorticity tends also to be convected along in the flow. Hence there are two transport mechanisms for vorticity, diffusion across fluid layers by viscosity and convection in the direction of flow. The latter mechanism dominates when a parameter called the Reynolds number, Re, is greater than unity ($Re = U\ell/\nu$ where U and ℓ are the typical velocity and dimension of the flow and ν is the kinematic viscosity of the fluid.) Then, as sketched in Figure 1, the vorticity will be confined to a thin layer near the solid surface, a region that is called the boundary layer. Beyond the trailing edge of the body the boundary layers form the wake behind the body. In contrast to the flow inside the boundary layer and wake, the flow outside tends to remain free of vorticity. Such a flow in which the vorticity is negligibly small is called an **irrotational flow** and is therefore characterized by

$$\underline{\omega} \;=\; \nabla(\underline{u}) \;=\; 0 \tag{Bdd4}$$

There are many circumstances and applications in which it is useful to solve for the details of an irrotational flow. One such class consists of flows in which solid boundaries play a minor role, for example waves on a deep ocean. Another class are flows at large Reynolds numbers in which the boundary layers are very thin compared with the other dimensions of the flow. Then, as illustrated in Figure 2, a first approximation would be to neglect the boundary layer and to solve the irrotational flow bounded by the geometry of the

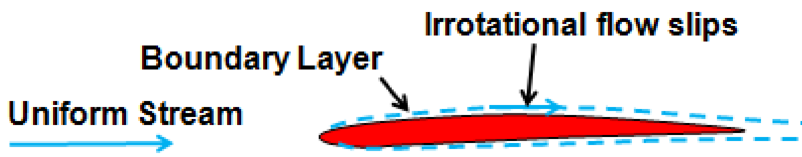

Figure 2: Thin boundary layer at large Reynolds numbers.

body instead of by the geometry of the outer surfaces of the boundary layers. Since the irrotational flow has a tangential velocity at the outer surfaces of the boundary layers, it would be inappropriate to apply the no-slip condition to that irrotational solution. But it would be appropriate to apply the condition of zero normal velocity. These and other details of such solutions are treated in more detail later in this text; the present intent is simply to indicate that irrotational flow solutions do have practical value in real fluid flows. Here we proceed to investigate some of the characteristics of irrotational flows and to develop a number of solution methodologies and examples. To do so we first introduce the **velocity potential**.