Method of Complex Variables for Planar Potential Flows

We have established that if we denote the complex potential, $\phi+i\psi$, by f and the complex position vector, $x+iy$, by z then **any** function, $f(z)$, corresponds to a particular planar potential flow and, furthermore, that the derivative $df/dz = u-iv$. Here we will begin with a few simple examples by choosing various $f(z)$ and determining the flows to which those functions correspond. Clearly $f(z) = Uz$ where U is a constant corresponds to a uniform stream in the x direction since then $df/dz = U$ therefore $u-iv = U$ and so u=U and $v = 0$. Similarly , $f(z) = (U-iV)z$ corresponds to a more general uniform stream with $u = U$ and $v = V$.

Introducing polar coordinates (r, θ) such that $x = \cos\theta$ and $y = \sin\theta$ it follows that

$$z \;=\; x+iy \;=\; re^{i\theta} \qquad \text{(Bgeb1)}$$

and therefore the particular function

$$f(z) \;=\; z^n \;=\; r^n e^{in\theta} \;=\; r^n(\cos n\theta + i\sin n\theta) \qquad \text{(Bgeb2)}$$

so that

$$\phi = r^n \;\cos n\theta \;,\; \psi = r^n \;\sin n\theta \;,\; u = nr^{n-1} \;\cos(n-1)\theta \;,\; v = -nr^{n-1} \;\sin(n-1)\theta \qquad \text{(Bgeb3)}$$

and this clearly corresponds to one of the components of the general solution to planar potential flow generated earlier by the method of separation of variables in polar coordinates. Moreover, the choice of the particular function, $f(z) = z^{-n}$ leads to the flow

$$\phi = r^{-n} \;\cos n\theta \;,\; \psi = -r^{-n} \;\sin n\theta \;,\; u = -nr^{-(n+1)} \;\cos(n+1)\theta \;,\; v = -nr^{-(n+1)} \;\sin(n+1)\theta \qquad \text{(Bgeb4)}$$

which corresponds to another group of terms in the aforementioned general solution. In addition there are two special cases in the same category, The first is the function, $f(z) = \ln z$ which corresponds to part of the special case in the aforementioned general solution, namely

$$\phi = \ln r \;,\; \psi = \theta \;,\; u = r^{-1} \;\cos\theta \;,\; v = r^{-1} \;\sin\theta \qquad \text{(Bgeb5)}$$

and is clearly a simple line source/sink. Secondly, the function $f(z) = -i\ln z$ corresponds to the other part of the special case in the aforementioned general solution, namely

$$\phi = \theta \;,\; \psi = -\ln r \;,\; u = -r^{-1} \;\sin\theta \;,\; v = r^{-1} \;\cos\theta \qquad \text{(Bgeb6)}$$

which is clearly a free vortex.

Before leaving the case of $f(z) = z^n$ and the flow it corresponds to given by equation (Bgeb3) we note that this is physically the flow in a wedge-shaped region of included angle π/n as shown in Figure 1. Thus $n = 2$ correspond to the flow in 45^o wedge region, $n = 3$ for a 30^o wedge region, etc. To verify this note that the velocity in the θ direction, u_θ, is

$$u_\theta \;=\; \frac{1}{r}\frac{\partial\phi}{\partial r} \;=\; -nr^{n-1}\sin n\theta \qquad \text{(Bgeb6)}$$

which means that $u_\theta = 0$ on the radial lines, $\theta = 0$ and $\theta = \pi/n$. These are therefore streamlines which can, in potential flow, be replaced by walls to yield the flow in the wedge between them (and, of course in

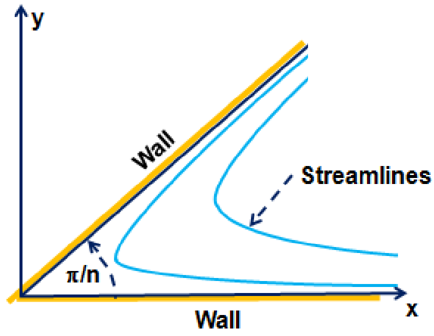

Figure 1: Planar potential flow in wedge-shaped region.

other wedges, with repeated flows at larger angles). Note that the velocities trend to zero in the apex of the wedge at $r \rightarrow 0$ (provided $n > 1$).

In contrast the case of $f(z) = z^{-n}$ (and the flow it corresponds to given by equation (Bgeb4)) is physically the potential flow around a wedge that projects into the fluid as sketched in Figure 2. To verify this note that

$$u_\theta \;=\; \frac{1}{r}\frac{\partial \phi}{\partial r} \;=\; -nr^{-(n+1)} \sin n\theta \tag{Bgeb7}$$

and therefore $u_\theta = 0$ on the radial lines, $\theta = 0$ and $\theta = \pi/n$. In this case the potential flow is outside the wedge formed by these radial lines (provided $n > 0$). Note that the velocities trend to infinity at the apex of the wedge at $r \rightarrow 0$ (provided $n > 0$).

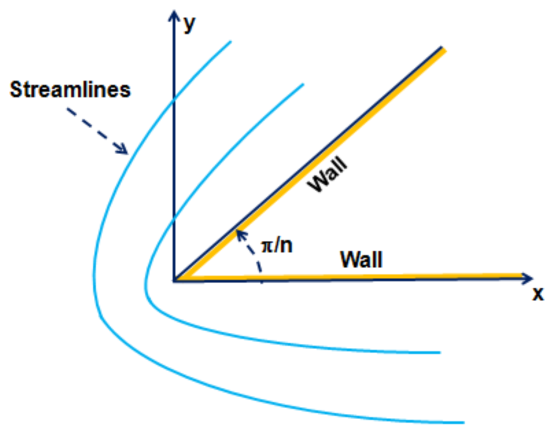

Figure 2: Planar potential flow around a wedge.

Among the potential flows of the type described by equation (Bgeb4) is that due to the function $f(z) = C/z$ where C is a constant which, for present purposes, we choose to be real. This yields

$$\phi = \frac{C\cos\theta}{r} \;,\; \psi = -\frac{C\sin\theta}{r} \;,\; u = -\frac{C\cos 2\theta}{r^2} \;,\; v = -\frac{C\sin 2\theta}{r^2} \tag{Bgeb8}$$

and can be recognized as that due to a doublet oriented in the x direction. Combining that with a uniform stream of velocity, U, and setting the constant C equal to UR^2 yields the flow of a uniform stream past a cylinder of radius, R. Thus the complex variable form of the solution for potential flow past a cylinder is

$$f(z) \;=\; U\left\{z + \frac{R^2}{z}\right\} \tag{Bgeb9}$$

The simplicity of this expression is an indication of how powerful the method of complex variables can be for planar potential flows. Furthermore we can add some circulation to the flow around a cylinder by adding a free vortex with $f(z) = -i(\Gamma/2\pi)\ln z$.

As a last example and for future purposes, we note that the flow generated at the position z by a series of vortex sheet elements of strength, γ_j ($j = 1$ to J), length s_j and location z_j can now be written compactly as

$$f(z) = \sum_{j=1}^{J} -\frac{i\gamma_j\ s_j}{2\pi} \ln(z - z_j) \tag{Bgeb10}$$

so that the velocities can be written as

$$\frac{df}{dz} = u - iv = \sum_{j=1}^{J} -\frac{i\gamma_j\ s_j}{2\pi(z - z_j)} \tag{Bgeb11}$$

Before proceeding to any more complicated examples, we need to introduce another major tool in creating planar potential flow solutions, namely the procedure of conformal mapping.