Gravity Wave Boundary Conditions

Since the form of the solution to Laplace's equation obtained by separation of variables in Cartesian coordinates, namely

$$\phi = (C_1 \sin kx + C_2 \cos kx)(C_3 e^{ky} + C_4 e^{-ky}) \quad \text{(Bgcb1)}$$

$$u = \frac{\partial \phi}{\partial x} = k(C_1 \cos kx - C_2 \sin kx)(C_3 e^{ky} + C_4 e^{-ky}) \quad \text{(Bgcb2)}$$

$$v = \frac{\partial \phi}{\partial y} = k(C_1 \sin kx + C_2 \cos kx)(C_3 e^{ky} - C_4 e^{-ky}) \quad \text{(Bgcb3)}$$

is oscillatory in the x direction and contains the arbitrary functions of time C_1, C_2, C_3 and C_4 it seems natural to use it to find solutions for the potential flow associated with gravity waves on an otherwise horizontal liquid surface. We will use the notation indicated in Figure 1. As usual the y coordinate is

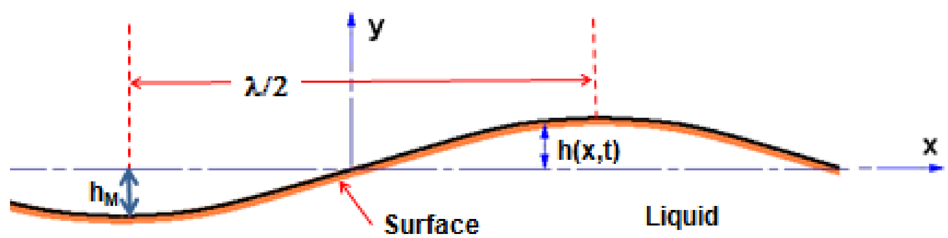

Figure 1: Notation used for gravity waves on an otherwise horizontal liquid surface.

vertically upward opposite to the direction of gravity, g. The mean position of the liquid surface is chosen to be $y = 0$ and the height of the waves is denoted by $h(x,t)$ where the maximum height is h_M. The wavelength of the waves is λ. Usually we will confine our analyses to small amplitude waves for which $h_M \ll \lambda$.

There are, of course, a number of different types of waves that one might encounter. **Standing waves** do not propagate in any direction, have fixed nodal locations on the x axis spaced a distance λ apart where the surface elevation, h, is always zero and other locations halfway between the nodes at which the surface elevation has its maximum amplitude, h_M. Consequently standing waves have a surface elevation given by

$$h(x,t) = h_M \sin \omega t \sin kx \quad \text{(Bgcb4)}$$

where ω is the frequency of the waves (we also define a wave period equal to $2\pi/\omega$) and clearly the wavenumber, $k = 2\pi/\lambda$. Note that we could equally well have used $\cos \omega t$ rather than $\sin \omega t$ and $\cos kx$ instead of $\sin kx$. However, we are free to choose the origin of both t and x and therefore the form of the relation (Bgcb4) is sufficiently general for present purposes.

Traveling waves on the other hand maintain a constant amplitude and shape but propagate in either the positive or negative x direction. Consequently traveling waves propagating in the positive x direction would have a surface elevation given by

$$h(x,t) = h_M \sin(kx - \omega t) \quad \text{(Bgcb5)}$$

and waves propagating in the negative x direction would have a surface elevation given by

$$h(x,t) = h_M \sin(kx + \omega t) \quad \text{(Bgcb6)}$$

Note that if we superimpose waves travelling in the positive x direction on waves of the same amplitude travelling in the negative x direction by adding together the expressions (Bgcb5) and (Bgcb6), the result is standing waves of amplitude $2h_M$.

Examining the forms of the waves given by the relations (Bgcb4), (Bgcb5) or (Bgcb6) and comparing these with the form of the solution, (Bgcb1), (Bgcb2) and (Bgcb3) it is clear that we could choose the functions of time $C_1(t)$ and $C_2(t)$ to simulate any of the three wave types and we will do this in the examples which follow.

First, however, we need to establish the form of the boundary conditions that should be applied at the liquid surface. Recall that, for potential flow, only one boundary condition is needed at a solid boundary, namely the condition of zero velocity normal to the boundary. However, the location of that solid boundary is normally known and given. In contrast the location of a "free" liquid surface usually has to be found as a part of the solution of the flow. That requires and additional boundary condition. Consequently, two boundary conditions are needed at a free liquid surface to solve for the associated potential flow. [Note that the situation is different for the Navier-Stokes equations where three additional boundary conditions are needed.] The two boundary conditions needed at a free surface of a potential flow are known as the **kinematic** and **dynamic** boundary conditions. The kinematic condition is associated with the kinematic relation between the liquid velocities and the rate of change of position of the free surface. On the other hand the dynamic condition is derived from the relation between the pressure in the overlying gas and that in the liquid at the surface.

In general, the kinematic boundary condition at a liquid free surface states that the liquid velocity normal to the surface at the surface must be equal to the rate of change of position of the surface in the normal direction (if there is evaporation or condensation this needs to be modified but we neglect this complication for present purposes). In the case of the small amplitude waves shown in Figure 1, the application of this condition to first order leads to the relation

$$\frac{\partial h}{\partial t} = (v)_{y=h} \tag{Bgcb7}$$

where, to first order, it is sufficient to evaluate v at $y = 0$ rather than at $y = h$; indeed, it can be shown that provided $h_M \ll \lambda$ then the second order corrections to (Bgcb4) are small. The present treatment will assume small amplitudes so that the kinematic boundary condition becomes

$$\frac{\partial h}{\partial t} = (v)_{y=0} \tag{Bgcb8}$$

The general form of the dynamic condition at a liquid free surface states that the normal and shear stresses in the gas (or other immiscible liquid) above the liquid surface must be equal to those in the liquid below the free surface (except, perhaps, for differences due to surface tension or surface rheology effects). In the present treatment we will assume for simplicity that the stress stae in the gas is simple, namely a constant or atmospheric pressure that we will denoted by p_a. In most cases we will also neglect any surface tension effects though a specific later example will examine those effects. Consequently our simplified dynamic boundary condition will be that the pressure in the liquid at the free surface be constant and equal to p_a. Since we are assuming potential flow and since Bernoulli's equation therefore applies it follows that the dynamic condition is that

$$\left\{\rho\frac{\partial \phi}{\partial t} + \rho\frac{|u|^2}{2} + \rho g y\right\}_{y=h} = \text{constant} \tag{Bgcb9}$$

Since the liquid is assume incompressible the ρ can be absorbed in the constant. Moreover, it can be shown that, in the examples delineated in this section, the $|u|^2$ is a second order term so that the dynamic condition becomes

$$\left\{ \frac{\partial \phi}{\partial t} \right\}_{y=h} + gh \;=\; \text{constant} \tag{Bgcb10}$$

and, finally, that the error in evaluating the first term at $y = 0$ rather than $y = h$ is also second order so the final form of the dynamic condition is

$$\left\{ \frac{\partial \phi}{\partial t} \right\}_{y=0} + gh \;=\; \text{constant} \tag{Bgcb11}$$

In studying gravity waves in the sections which follow the forms of the kinematic and dynamic boundary conditions used will be equations (Bgcb8) and (Bgcb11) unless otherwise noted.