Gravity Waves on an Infinitely Deep Ocean

To construct the solution for waves on an infinitely deep ocean we first note that the terms in equations (Bgca6) to (Bgca8) which involve e^{-ky} must vanish otherwise the velocities would grow without limit at large depths ($y \to -\infty$). Therefore C_4 must be zero and then C_3 can be absorbed into C_1 and C_2 so that

$$\phi = (C_1 \sin kx + C_2 \cos kx)e^{ky} \qquad \text{(Bgcc1)}$$

$$u = \frac{\partial \phi}{\partial x} = k(C_1 \cos kx - C_2 \sin kx)e^{ky} \qquad \text{(Bgcc2)}$$

$$v = \frac{\partial \phi}{\partial y} = k(C_1 \sin kx + C_2 \cos kx)e^{ky} \qquad \text{(Bgcc3)}$$

As before we will use the notation indicated in Figure 1. where $k = 2\pi\lambda$ and the wave amplitude is small,

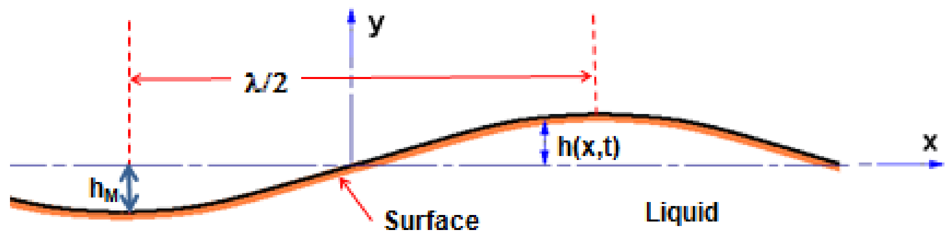

Figure 1: Notation used for gravity waves on an infinitely deep ocean.

$h_M \ll \lambda$.

We choose to first examine travelling waves propagating in the positive x direction so that the surface elevation must be of the form

$$h(x,t) = h_M \sin(kx - \omega t) \qquad \text{(Icb2)}$$

where ω is the wave frequency. Using the kinematic boundary condition at the free surface, namely

$$\frac{\partial h}{\partial t} = (v)_{y=0} \qquad \text{(Icb5)}$$

it follows that

$$(v)_{y=0} = \frac{\partial h}{\partial t} = -h_M \omega \cos(kx - \omega t) \qquad \text{(Bgcc4)}$$

Comparing this with the expression for $v_{y=0}$ that follows from equation (Bgcc3) namely

$$(v)_{y=0} = k(C_1 \sin kx + C_2 \cos kx) \qquad \text{(Bgcc5)}$$

it must follow that $C_1(t)$ and $C_2(t)$ for this particular case must be

$$C_1(t) = -\frac{h_M \omega}{k} \sin \omega t \quad \text{and} \quad C_2(t) = -\frac{h_M \omega}{k} \cos \omega t \qquad \text{(Bgcc6)}$$

so that the solution to the potential flow is

$$\phi = -\frac{h_M \omega}{k} \cos(kx - \omega t)e^{ky} \qquad \text{(Bgcc7)}$$

$$u = \frac{\partial \phi}{\partial x} = +h_M \omega \sin(kx - \omega t) e^{ky} \qquad \text{(Bgcc8)}$$

$$v = \frac{\partial \phi}{\partial y} = -h_M \omega \cos(kx - \omega t) e^{ky} \qquad \text{(Bgcc9)}$$

The final step in the solution is to apply the dynamic condition at the liquid surface namely

$$\left\{ \frac{\partial \phi}{\partial t} \right\}_{y=0} + gh = \text{constant} \qquad \text{(Icb8)}$$

and this yields the expression

$$-\frac{h_M \omega^2}{k} \sin(kx - \omega t) + g h_M \sin(kx - \omega t) = \text{constant} \qquad \text{(Bgcc10)}$$

which can only be satisfied if

$$\omega^2 = gk \quad \text{or} \quad \omega = (gk)^{\frac{1}{2}} = (2\pi g/\lambda)^{\frac{1}{2}} \qquad \text{(Bgcc11)}$$

In terms of the wave frequency, f, in Hertz (ω is in radians/sec and therefore $f = \omega/2\pi$)

$$f = (g/2\pi\lambda)^{\frac{1}{2}} \qquad \text{(Bgcc12)}$$

and the speed of propagation of the wave, $c = \omega/k$, is

$$c = (g/k)^{\frac{1}{2}} = (g\lambda/2\pi)^{\frac{1}{2}} \qquad \text{(Bgcc13)}$$

Typical ocean waves with λ of about $10m$ therefore propagate at about $c = 4m/s$. However, very large wavelength tsunami waves with typical wavelength of the order of $1000m$ travel at velocities of about $40m/s$ or $100miles/hr$. Thus tsunami waves travel great distances across the Pacific basin in a matter of a day or less.

Finally we note that we can write down the solution to waves propagating in the negative rather than positive x direction simply by reversing the sign of c in equations (Bgcc7) to (Bgcc9).