Method of Separation of Variables in Cartesian Coordinates

Prior to addressing the potential flow solutions for gravity waves, we will first establish the form of the solutions to Laplace's equation

$$\nabla^2\phi \;=\; 0 \quad \text{or} \quad \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \tag{Bgca1}$$

for planar flow that result from using the method of the separation of variables in Cartesian coordinates. We seek a separable solution of the form

$$\phi \;=\; X(x,t)\;Y(y,t) \tag{Bgca2}$$

where the functions $X(x,t)$ and $Y(y,t)$ need to be determined. Substituting this into equation (Bgca1) and rearranging yields

$$\frac{1}{X}\frac{d^2X}{dx^2} \;=\; -\frac{1}{Y}\frac{d^2Y}{dy^2} \tag{Bgca3}$$

and since the left hand side is a function only of x and t and the right hand side is a function only of y and t both sides can only be a function of t. Here we choose to set them both equal to a negative constant, $-k^2$, so that

$$\frac{d^2X}{dx^2} \;=\; -k^2X \quad \text{and} \quad \frac{d^2Y}{dy^2} \;=\; k^2Y \tag{Bgca4}$$

and these two ordinary differential equations have the following solutions

$$X \;=\; C_1\sin kx + C_2\cos kx \quad \text{and} \quad Y \;=\; C_3e^{ky}+C_4e^{-ky} \tag{Bgca5}$$

where the quantities C_1, C_2, C_3, and C_4, may be constants or functions of time. Hence the form of the solution obtained by this methodology is

$$\phi \;=\; (C_1\sin kx + C_2\cos kx)(C_3e^{ky}+C_4e^{-ky}) \tag{Bgca6}$$

$$u \;=\; \frac{\partial\phi}{\partial x} \;=\; k(C_1\cos kx - C_2\sin kx)(C_3e^{ky}+C_4e^{-ky}) \tag{Bgca7}$$

$$v \;=\; \frac{\partial\phi}{\partial y} \;=\; k(C_1\sin kx - C_2\cos kx)(C_3e^{ky}-C_4e^{-ky}) \tag{Bgca8}$$

Several asides are appropriate at this point. Firstly, note that we have reduced the problem to two ordinary differential equations instead of one partial differential equation; this is the essence of the method of separation of variables. Secondly, note that the desired solution and/or the boundary conditions that have to be applied may not be compatible with the form of the solution that emerged using this approach. If so, then some other approach is needed. Thirdly, note that if we had chosen to equate the two sides of equation (Bgca3) to a positive constant rather than a negative one then the solution would have been oscillatory in the y direction and exponential in the x direction rather than the other way around. Such a solution is equally valid and has application in other contexts but we choose to focus here on the solution represented by equations (Bgca6) to (Bgca8) which are useful in studying gravity waves on a fluid surface.