Gravity Waves on an Finite-Depth Ocean

The solution of the potential flow for traveling waves on an ocean of finite depth is very similar to that for an infinitely deep ocean except for the boundary condition on the ocean bottom at $y = -H$ as sketched in Figure 1. Beginning with the potential flow solution of equations (Bgca6) to (Bgca8) we note that since

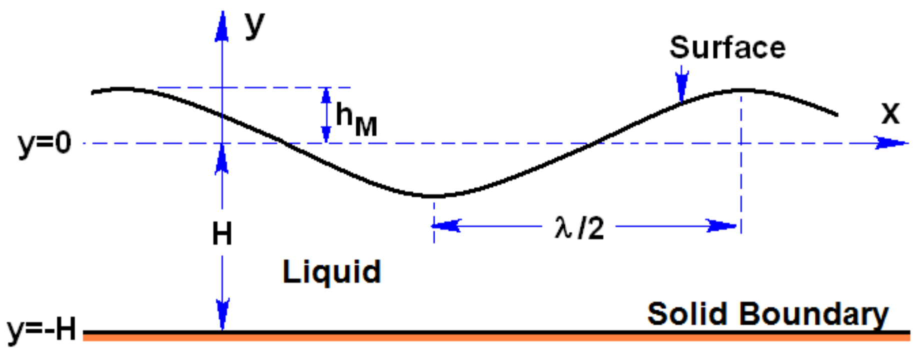

Figure 1: Notation used for gravity waves on an ocean of finite depth, H.

the velocity, v, must be zero at $y = -H$ it follows that

$$C_3 e^{-kH} - C_4 e^{kH} = 0 \quad \text{so that} \quad C_4 = C_3 e^{-2kH} \tag{Bgcd1}$$

and, after substituting into equations (Bgca6) to (Bgca8), we can absorb the remaining constant into C_1 and C_2 and write the solution as

$$\phi = (C_1 \sin kx + C_2 \cos kx) \cosh k(y+H) \tag{Bgcd2}$$

$$u = \frac{\partial \phi}{\partial x} = k(C_1 \cos kx - C_2 \sin kx) \cosh k(y+H) \tag{Bgcd3}$$

$$v = \frac{\partial \phi}{\partial y} = k(C_1 \sin kx + C_2 \cos kx) \sinh k(y+H) \tag{Bgcd4}$$

where, as before, $k = 2\pi\lambda$ and the wave amplitude is small, $h_M \ll \lambda$.

We choose to examine traveling waves propagating in the positive x direction so that the surface elevation must be of the form

$$h(x,t) = h_M \sin(kx - \omega t) \tag{Bgcd5}$$

where ω is the wave frequency. Using the kinematic boundary condition at the free surface, namely

$$\frac{\partial h}{\partial t} = (v)_{y=0} \tag{Bgcd6}$$

it follows that

$$(v)_{y=0} = \frac{\partial h}{\partial t} = -h_M \omega \cos(kx - \omega t) \tag{Bgcd7}$$

Comparing this with the expression for $v_{y=0}$ that follows from equation (Bgcd4) namely

$$(v)_{y=0} = k(C_1 \sin kx + C_2 \cos kx) \sinh kH \tag{Bgcd8}$$

it must follow that $C_1(t)$ and $C_2(t)$ for this particular case must be

$$C_1(t) = -\frac{h_M\omega}{k\sinh kH}\sin\omega t \quad \text{and} \quad C_2(t) = -\frac{h_M\omega}{k\sinh kH}\cos\omega t \tag{Bgcd9}$$

so that the solution to the potential flow is

$$\phi = -\frac{h_M\omega}{k\sinh kH}\cos(kx-\omega t)\cosh k(y+H) \tag{Bgcd10}$$

$$u = \frac{\partial\phi}{\partial x} = \frac{h_M\omega}{\sinh kH}\sin(kx-\omega t)\cosh k(y+H) \tag{Bgcd11}$$

$$v = \frac{\partial\phi}{\partial y} = -\frac{h_M\omega}{\sinh kH}\cos(kx-\omega t)\sinh k(y+H) \tag{Bgcd12}$$

The final step in the solution is to apply the dynamic condition at the liquid surface namely

$$\left\{\frac{\partial\phi}{\partial t}\right\}_{y=0} + gh = \text{constant} \tag{Bgcd13}$$

and this yields the expression

$$-\frac{h_M\omega^2}{k\sinh kH}\sin(kx-\omega t)\cosh kH + gh_M\sin(kx-\omega t) = \text{constant} \tag{Bgcd14}$$

which can only be satisfied if

$$\omega^2 = gk\tanh kH \quad \text{or} \quad \omega = (gk\tanh kH)^{\frac{1}{2}} \tag{Bgcd15}$$

In terms of the wave frequency, f, in Hertz (ω is in radians/sec and therefore $f = \omega/2\pi$)

$$f = (g\tanh(2\pi H/\lambda)/2\pi\lambda)^{\frac{1}{2}} \tag{Bgcd16}$$

and the speed of propagation of the wave, $c = \omega/k$, is

$$c = (g\tanh(kH)/k)^{\frac{1}{2}} = (g\lambda\tanh(2\pi H/\lambda)/2\pi)^{\frac{1}{2}} \tag{Bgcd17}$$

and therefore the speed of the waves decreases as the depth decreases. This is readily observed at the beach. Waves encountering shallower depths as they approach the beach, slow down and, as a result, tend to pile up and break. Note for future use, that when $\lambda \ll H$, the wave speed, c, tends to

$$c = (gH)^{\frac{1}{2}} \tag{Bgcd18}$$

Extensive use will be made of this in later sections on open channel flow.

Finally we note that we can write down the solution to waves propagating in the negative rather than positive x direction simply by reversing the sign of ω in equations (Bgcd10) to (Bgcd12).