Cylinder with Circulation in a Uniform Stream

If we now add a free vortex at the origin to the flow of a uniform stream past a doublet we modify the

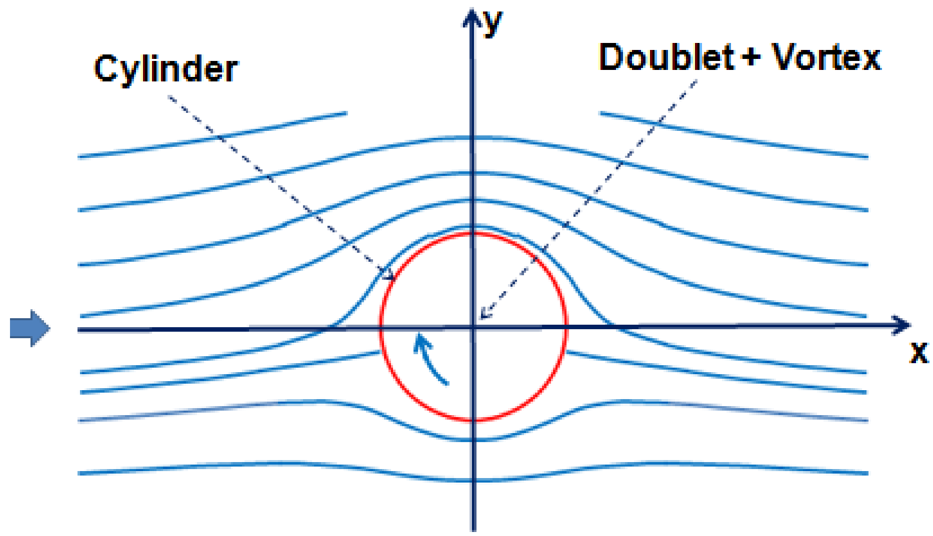

Figure 1: Streamlines in the potential flow of a doublet plus a vortex in a uniform stream.

flow field as seen in Figure 1. This planar potential flow therefore has the velocity potential,

$$\phi = U\left[r + \frac{R^2}{r}\right]\cos\theta + \frac{\Gamma}{2\pi}\theta \tag{Bgdi1}$$

where the circulation, Γ, is considered positive in the anticlockwise direction (Figure 1 shows a negative circulation) and the velocities are

$$u_r = U\left[1 - \frac{R^2}{r^2}\right]\cos\theta \tag{Bgdi2}$$

$$u_\theta = -U\left[1 + \frac{R^2}{r^2}\right]\sin\theta + \frac{\Gamma}{2\pi r} \tag{Bgdi3}$$

where the polar coordinates, r and θ, are $x = r\cos\theta$ and $y = r\sin\theta$ as before. This simulates the flow pattern associated with a spinning cylinder. Notice that the flow velocities above and below the cylinder are no longer the same as they were without the vortex. Indeed the stagnation points are no longer located at $\theta = 0, \pi$ but at

$$\arcsin\theta = \frac{\Gamma}{4\pi UR} \tag{Bgdi4}$$

It is of interest to evaluate the force on the cylinder in the y direction which this asymmetry creates. Since the velocity on the surface of the cylinder is

$$(u_\theta)_{r=R} = -2U\sin\theta + \frac{\Gamma}{2\pi R} \tag{Bgdi5}$$

it follows from Bernoulli's equation that the pressure, p, on the surface is given by

$$(p)_{r=R} \;=\; p_\infty + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho\left[\frac{\Gamma}{2\pi R} - 2U\sin\theta\right]^2 \qquad \text{(Bgdi6)}$$

and the force on the cylinder (per unit dimension normal to the plane of the flow), L, in the y direction will be given by

$$L \;=\; -\int_0^{2\pi}(p)_{r=R}R\sin\theta d\theta \qquad \text{(Bgdi7)}$$

Substituting for $(p)_{r=R}$ and integrating yields the classical result that

$$L \;=\; -\rho U\Gamma \qquad \text{(Bgdi8)}$$

We shall see that this result holds for any finite body with a circulation, Γ, that is placed in a uniform stream, U. The lift is, of course, produced by the fact that the surface velocity on one side of the cylinder is reduced by the circulation while that on the other side is increased. This produces an increase in the pressure on one side and a decrease in pressure on the other side; hence the lift force. This is known as the **Magnus Force** and, in practice, occurs with a spinning cylinder or ball provided the Reynolds number is not too small. It is the force that effects the flight of a spinning golf ball, baseball or soccer ball. Even helicopter blades consisting of spinning cylinders have been produced.