Images and Walls

As exemplified in the section on Kelvin ovals, we can further supplement the method of singularities by using image singularities to create planes of symmetry that can then be replaced by solid boundaries in order to simulate flows that involve walls. This methodology is based on the fact that, in potential flow, the only boundary condition that is required at a solid boundary is that of zero velocity normal to that boundary. Therefore, any streamline can be replaced by a solid boundary or wall. This feature is in marked contrast to a real viscous flow in which both the zero normal velocity condition and the no-slip (or zero tangential velocity) condition need to be satisfied at a solid boundary. Therefore, in a real viscous flow, while a solid boundary is always a streamline, only a few very special streamlines can be replaced by solid boundaries.

A few more examples will further illustrate the method of images and some of the details that require attention in implementing this technique. As a first example we will analyze the case of Kelvin ovals a little further. Note first that the image vortex has to be of opposite sign in order to satisfy $v_{y=0} = 0$. Also, visualizing the potential flow as that due to a vortex of ciculation, Γ, a distance a from a wall we can use this as a way of qualitatively determining the effect of a nearby wall on the lift generated by an airfoil with that circulation. To do so we first evaluate the pressure, p, on the wall. Starting with the velocity, $u_{y=0}$ which, from equation (Bgdk2), is

$$u_{y=0} \; = \; U - \frac{\Gamma a}{\pi[x^2 + a^2]} \tag{Bgdm1}$$

Bernoulli's equation yields the pressure

$$p_{y=0} - p_\infty \; = \; \frac{\rho}{2}\left[\frac{2\Gamma U a}{\pi(x^2+a^2)} - \frac{\Gamma^2 a^2}{\pi^2(x^2+a^2)^2}\right] \tag{Bgdm2}$$

where p_∞ is the pressure far from the region of interest. Integrating this pressure over the entire length of the wall, we can evaluate the force, F_y, on the wall in the positive y direction per unit depth normal to the plane of the flow as

$$F_y \; = \; \int_{-\infty}^{+\infty} (p_{y=0} - p_\infty)\; dx \; = \; \rho\Gamma U \left[1 - \frac{\Gamma}{4\pi U a}\right] \tag{Bgdm3}$$

where the value of Γ in the Kelvin oval construction was negative. Moreover, if we visualize the vortex as a rudimentary airfoil at a positive angle of incidence so that we expect it to experience lift in the positive y direction then $(-\Gamma)$ is a positive number. If we invoke the momentum theorem for a control volume consisting of the entire fluid domain minus an infinitesmal region around the center of the vortex, it follows that the force on the vortex or rudimentary airfoil must be equal and opposite to the force on the wall since the net momentum flux out of the control volume is zero. Therefore the lift L experienced by the vortex or rudimentary airfoil per unit depth normal to the plane of the flow in the positive y direction is

$$L \; = \; -F_y \; = \; \rho(-\Gamma)U\left[1 + \frac{(-\Gamma)}{4\pi U a}\right] \tag{Bgdm4}$$

When $a \to \infty$ we recover the expected expression for the lift, namely $\rho(-\Gamma)U$. However the above expression also demonstrates that the effect of a nearby wall is to increase the lift by a fractional amount equal to $(-\Gamma)/4\pi U a$. This result is only an estimate for the case when the distance from the wall is quite large; other singularities will be needed to simulate the details of the flow close to an airfoil and these will modify the above result.

A similar example would simulate the flow around a finite body close to a wall by inserting a doublet (rather than a vortex) at $x = 0$, $y = a$ and then inserting at image doublet with the same strength at $x = 0$, $y = -a$. Expanding on the construction of the flow around a cylinder (equations (Bgdh3) to (Bgdh5)) the velocity potential due to the uniform stream and the two doublets would be

$$\phi \;=\; Ux + \frac{Qa}{\pi}\frac{x}{(x^2+(y-a)^2)} + \frac{Qa}{\pi}\frac{x}{(x^2+(y+a)^2)} \qquad \text{(Bgdm5)}$$

By evaluating the velocities on the centerline wall, then obtaining the pressure using Bernoulli's equation and integrating to obtain the force on the wall in the y direction, one can find the force on the body that tends to force it away from the wall. This occurs because of the reduction in the velocity and therefore the increase in pressure in between the doublet-produced body and the wall. Note however that the closed streamlines that generate the body are no longer circular so the superposition does not simulate the flow around a circular cylinder near a wall but rather a distorted cylinder. This highlights one of the difficulties with the method of superposition of singularities, namely that the body shapes generated may not correspond with those desired. We will address this issue in other pages to come.

Consider next the flow around an object between two plane walls as sketched in Figure **??**. Then one must use images in each wall in a first effort to satisfy the zero normal velocity in each of the walls. But then images of those images in the other walls will clearly be needed. This process of trying to satisfy the two boundary conditions is endless and one will have to sum the effects of this infinite array of images in order to produce an accurate potential flow solution. Fortunately, the images are increasingly distant from the original body and walls and so the corrections are commonly convergent. But the mathematics can become impossibly complicated. There are however examples of multiple images which do not produce an infinite array. For example, we could simulate a source inserted into a corner flow by utilizing symmetric images in each of the four quadrants as sketched in Figure **??**.