Lift and Drag in Potential Flow

In this section we will derive general relations for the lift and drag forces in the steady, planar, incompressible potential flow around a finite body placed in a uniform stream of velocity, U. To do so we examine the flow at a very large radius, r, away from the finite body as depicted in Figure 1. At this large radius, we

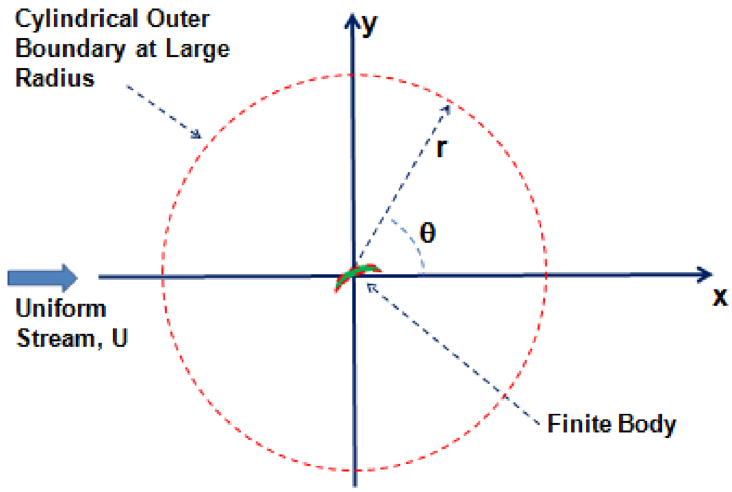

Figure 1: Finite body in a uniform stream.

expect that only the terms in the general potential flow solution (Bgdb10)to (Bgdb13) which decay least slowly will effect our analysis. Since the body is finite the net source term in equation (Bgdb11) must be zero which means that $C_{40}C_{10} = 0$ and if we eliminate the superfluous constant, retain the uniform stream in the x direction (but not the y direction), retain the free vortex term and the doublet and higher order terms the form of the velocity potential can be written as

$$\phi \;=\; Ux + \frac{\Gamma}{2\pi}\theta + \frac{f_1(\theta)}{r} + \frac{f_2(\theta)}{r^2} + O\left(r^{-3}\right) \tag{Bgdj1}$$

where the functions $f_1(\theta)$, $f_2(\theta)$, etc. maybe functions of the shape and size of the body. The associated velocities are

$$u_r \;=\; U\cos\theta - \frac{f_1(\theta)}{r^2} - \frac{2f_2(\theta)}{r^3} + O\left(r^{-4}\right) \tag{Bgdj2}$$

$$u_\theta \;=\; -U\sin\theta + \frac{\Gamma}{2\pi r} + \frac{1}{r^2}\frac{df_1(\theta)}{d\theta} + \frac{1}{r^3}\frac{df_2(\theta)}{d\theta} + O\left(r^{-4}\right) \tag{Bgdj3}$$

Now apply the momentum theorem on a control volume with an outer circular boundary at some large radius r and excluding the body itself near $r = 0$. First we evaluate the forces acting on this control volume. Neglecting any effect due to gravity we can use Bernoulli's equation to find an expression for the pressure, p, acting on the large circular outer boundary of the control volume. This yields

$$(p)_{r=\infty} + \frac{\rho}{2}U^2 \;=\; (p)_{r=r} + \frac{\rho}{2}\left[\left\{U\cos\theta - \frac{f_1(\theta)}{r^2} - \frac{2f_2(\theta)}{r^3} + O\left(r^{-4}\right)\right\}^2 + \left\{-U\sin\theta + \frac{\Gamma}{2\pi r} + \frac{1}{r^2}\frac{df_1(\theta)}{d\theta} + \frac{1}{r^3}\frac{df_2(\theta)}{d\theta} + \right.\right. \tag{Bgdj4}$$

so that

$$(p)_{r=r} \;=\; (p)_{r=\infty} + \frac{\rho U \Gamma}{2\pi r}\sin\theta - \frac{\rho}{2}\left\{\frac{\Gamma}{2\pi r}\right\}^2 + \frac{\rho U}{r^2}\left\{f_1(\theta)\cos\theta + \frac{df_1(\theta)}{d\theta}\sin\theta\right\} + O\left(r^{-3}\right) \qquad \text{(Bgdj5)}$$

Integrating the pressure around the outer circular boundary to determine the forces (per unit depth normal to the plane of the flow) due to the pressure, $(p)_{r=r}$, in the the x and y directions, F_{px} and F_{py}, we find

$$F_{px} \;=\; -\int_0^{2\pi} pr\cos\theta d\theta \;=\; O\left(r^{-1}\right) \qquad \text{(Bgdj6)}$$

$$F_{py} \;=\; -\int_0^{2\pi} pr\sin\theta d\theta \;=\; -\frac{\rho U \Gamma}{2} + O\left(r^{-1}\right) \qquad \text{(Bgdj7)}$$

If the drag force (per unit depth normal to the plane of the flow) on the body in the positive x direction (the direction of the uniform stream) is denoted by D and the lift force (per unit depth normal to the plane of the flow) on the body in the positive y direction is denoted by L then the total forces, F_x and F_y on the fluid in the control volume in the positive directions are

$$F_x - D \;=\; -D + O\left(r^{-1}\right) \qquad \text{(Bgdj8)}$$

$$F_y \;=\; -L - \frac{\rho U \Gamma}{2} + O\left(r^{-1}\right) \qquad \text{(Bgdj9)}$$

We now use the momentum theorem to equate these net forces to the fluxes of momentum out of the control volume (since the flow is steady there is no change of momentum within the control volume). The fluxes at the inner surface of the control volume, the surface of the body, are zero since there is no mass flux across that surface. The flux of momentum in the x direction (per unit depth normal to the plane of the flow) out of the control volume through the outer boundary at large r will be

$$\int_0^{2\pi} (\rho u_r r d\theta)(u_r \cos theta - u_\theta \sin\theta) \qquad \text{(Bgdj10)}$$

and the corresponding flux in the y direction will be

$$\int_0^{2\pi} (\rho u_r r d\theta)(u_r \sin theta + u_\theta \cos\theta) \qquad \text{(Bgdj11)}$$

Substituting for the velocities from equations (Bgdj2) and (Bgdj3), integrating and using the momentum theorem to equate the results to the net forces, F_x and F_y, we obtain

$$D \;=\; O\left(r^{-1}\right) \qquad \text{(Bgdj12)}$$

$$L \;=\; -\rho U \Gamma + O\left(r^{-1}\right) \qquad \text{(Bgdj13)}$$

But since the radius of the outer boundary of the control volume was arbitrary, the error terms of order r^{-1} and higher must, in fact, be zero so that

$$D \;=\; 0 \quad \text{and} \quad L \;=\; -\rho U \Gamma \qquad \text{(Bgdj14)}$$

which are two important and classic results from the theory of steady, planar potential flow for the forces on a finite body in a uniform stream of velocity, U.

The first of these results, $D = 0$, is known as d'Alembert's Paradox and it can be extended to three-dimensional bodies without much difficulty. The fact that it was so at odds with practical experience puzzled the mathematicians of the late 19th century and it was not until the work of Prandtl at the beginning of the 20th century that the paradox was resolved. In fact the result is readily understood by recognizing that a body with finite drag would necessarily be doing work on the fluid at a rate of DU yet there is no mechanism in the fluid mechanics to dissipate that energy yet maintain a steady flow. As this suggest the answer to the paradox must involve dissipative mechanisms and we shall see later that the viscous effects are central to resolving the paradox.

The second of these results, namely $L = -\rho\Gamma U$, is a very useful and practical result which we will make much use of in later sections to explain and develop the lift on airfoils and other devices. Of course, as yet, it is unclear what determines the circulation, Γ, and this too must await later developments including analyses of the effects of viscosity. We earlier developed a special case of this result, namely the lift on a spinning cylinder in potential flow but the astute reader will have recognized that we did not explain how the circulation was transmitted to the fluid by the spinning cylinder. We only observed that if such circulation existed then the lift on the cylinder (known in that case as the Magnus Force) would be given by $L = -\rho\Gamma U$. However, it is an effect well known in practice through the effects of spin on the flight of golf balls, baseballs and soccer balls. Even helicopter blades consisting of spinning cylinders have been produced.