Planar Rankine Body

A line source and a line sink of equal strength placed some distance apart will produce the typical

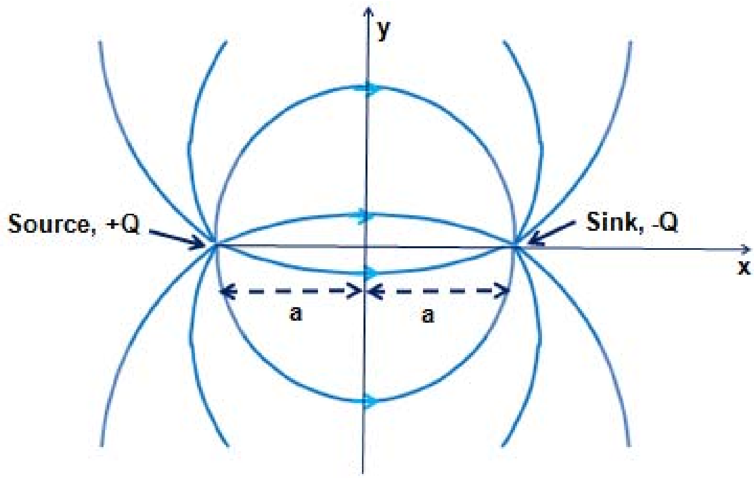

Figure 1: Streamlines due to a source and sink of equal strength.

streamline pattern shown in Figure 1. The sum of the potentials due to a line source at $x = -a$, $y = 0$ and a line sink at $x = a$, $y = 0$ will generate a velocity potential and streamfunction given by

$$\phi = \frac{Q}{4\pi}\ln\left\{(x+a)^2+y^2\right\} - \frac{Q}{4\pi}\ln\left\{(x-a)^2+y^2\right\} = \frac{Q}{4\pi}\ln\left\{\frac{(x+a)^2+y^2}{(x-a)^2+y^2}\right\} \quad \text{(Bgdg1)}$$

$$\psi = \frac{Q}{2\pi}\left[\arctan\left\{\frac{y}{(x+a)}\right\} - \arctan\left\{\frac{y}{(x-a)}\right\}\right] \quad \text{(Bgdg2)}$$

Then, if we add a uniform stream of velocity, U, the result is a velocity potential and streamfunction given by

$$\phi = Ux + \frac{Q}{4\pi}\ln\left\{\frac{(x+a)^2+y^2}{(x-a)^2+y^2}\right\} \quad \text{(Bgdg3)}$$

$$u = \frac{\partial\phi}{\partial x} = U + \frac{Qa}{\pi}\frac{\{y^2-(x^2-a^2)\}}{\{(x^2+y^2+a^2)^2-(2ax)^2\}} \quad \text{(Bgdg4)}$$

$$v = \frac{\partial\phi}{\partial y} = -\frac{2Qaxy}{\{(x^2+y^2+a^2)^2-(2ax)^2\}} \quad \text{(Bgdg5)}$$

$$\psi = Uy + \frac{Q}{2\pi}\left[\arctan\left\{\frac{y}{(x+a)}\right\} - \arctan\left\{\frac{y}{(x-a)}\right\}\right] \quad \text{(Bgdg6)}$$

and the pattern of streamlines shown in Figure 2. The red, closed streamline is then known as a planar Rankine body whose dimensions, b/a, and c/a can be calculated from equations (Bgdg4) to (Bgdg6) in a manner similar to the way we calculated the dimensions of a Rankine half-body. On $y = 0$ it follows from equation (Bgdg4) that

$$(u)_{y=0} = U - \frac{Qa}{\pi(x^2-a^2)} \quad \text{(Bgdg7)}$$

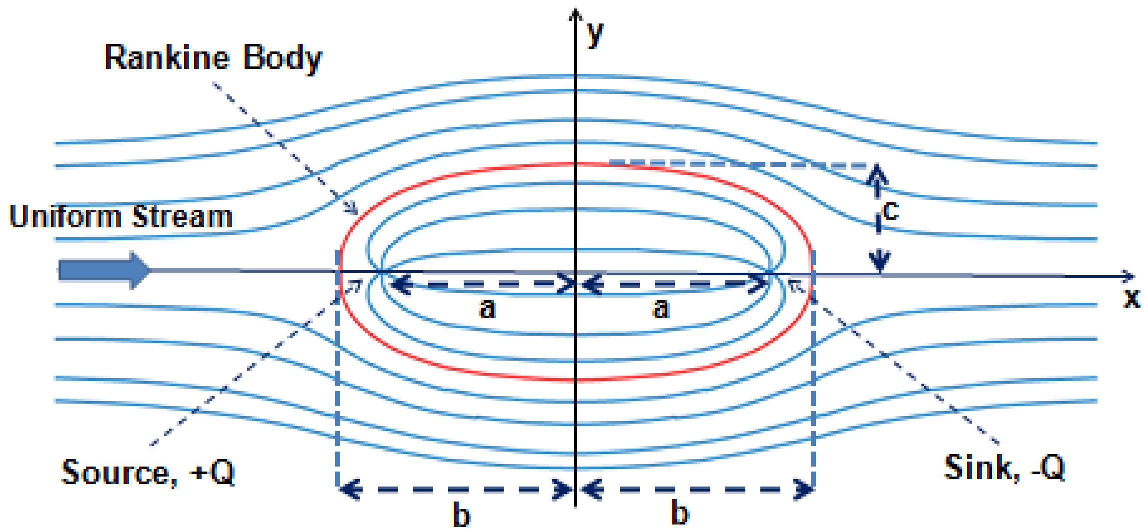

Figure 2: Planar potential flow of a source and sink of equal strength in a uniform stream showing streamlines and the Rankine body in red.

and this will be equal to zero at the front and rear stagnation points, $x = \pm b$, so it follows that the longitudinal dimension of the Rankine body is given by

$$\frac{b}{a} = \left[1 + \frac{Q}{4aU}\right]^{\frac{1}{2}} \qquad \text{(Bgdg8)}$$

Furthermore, since $\psi = 0$ when $y = 0$ it follows that the surface of the Rankine body must also have $\psi = 0$. Therefore the lateral dimension of the Rankine body is given by the point on the y axis at which $\psi = 0$, namely $y = \pm c$ and this means that

$$\frac{c}{a} = \frac{Q}{2Ua} - \frac{Q}{\pi Ua} \arctan \frac{c}{a} \qquad \text{(Bgdg9)}$$

equations which can be solved numerically to find c/a as a function of Q/Ua. Note that for long thin Rankine bodies such that $c/a/ll1$ it follows that $c/a \approx Q/2Ua$.

Some typical dimensions of these Rankine bodies or ”ovals” are listed in Table 1 as a function of the dimensionless parameter, Q/Ua. The last column lists the maximum velocity, u_{max}/U, at the equator or maximum radius of the oval. Note that this increases as the oval becomes more bluff, tending to 2 at large Q/Ua when the oval tends toward a cylinder.

TABLE 1. Shape of some Rankine ovals.

Q/Ua	c/a	b/a	b/c	u_{max}/U
0.0	0.0	1.0	∞	1.0
0.01	0.031	1.010	32.79	1.020
0.1	0.263	1.095	4.169	1.187
1.0	1.307	1.732	1.326	1.739
10.0	4.435	4.583	1.033	1.968
100.0	14.13	14.18	1.003	1.997
∞	∞	∞	1.000	2.000

Clearly an infinite array of objects and the flow around or through them can be simulated by the superposition of sources, sinks, doublets, vortices, etc. and this method of superposition can be extended to three-dimensions as will be seen in the pages that follow. There are a number of internet resources which allow one to do this and to examine the streamlines and bodies that are a part of those flows. We would encourage the student to explore the possibilities, for example by using the internet site called the "Ideal Flow Machine" at http://www.aoe.vt.edu/ devenpor/aoe5104/ifm/ifm.html.

There is one important feature of the above solution for potential flow around a finite body that requires comment. Note from equations (Bgdg4) and (Bgdg5) that the magnitudes of the velocities on the front of the body are identical to those at the mirror image point on the rear of the body. Therefore, from Bernoulli's equation it follows that the pressure distribution on the front of the body is identical to that over the rear. Consequently if we integrate the pressures in order to find the net force on the body in the x direction the result must be identical to zero. This disconcerting result which is so at odds with our practical experience is true for all potential flows around finite bodies as we demonstrate elsewhere. It is known as **D'Alembert's paradox**. Both the paradox and its practical resolution will be the subject of much discussion in future pages.