Planar Rankine Half-Bodies

If we superimpose a planar source on a uniform stream we can create streamlines which can be replaced by a solid body so as to generate the potential flow around that body. A simple example of this is the superposition of a source and a uniform stream. We choose a uniform stream of velocity U in the x direction so that its velocity potential contribution is Ux. Then, combined with a planar source of strength, Q, this generates a planar flow with the following features:

$$\phi = Ux + \frac{Q}{4\pi}\ln(x^2+y^2) = Ur\cos\theta + \frac{Q}{2\pi}\ln r \quad \text{(Bgde1)}$$

$$u_r = U\cos\theta + \frac{Q}{2\pi r} \quad ; \quad u_\theta = U\sin\theta \quad \text{(Bgde2)}$$

$$\psi = Ur\sin\theta + \frac{Q\theta}{2\pi} \quad \text{(Bgde3)}$$

where $x = r\cos\theta$ and $y = r\sin\theta$. This combination produces the streamlines sketched in Figure 1 whose detailed geometry we will now explore.

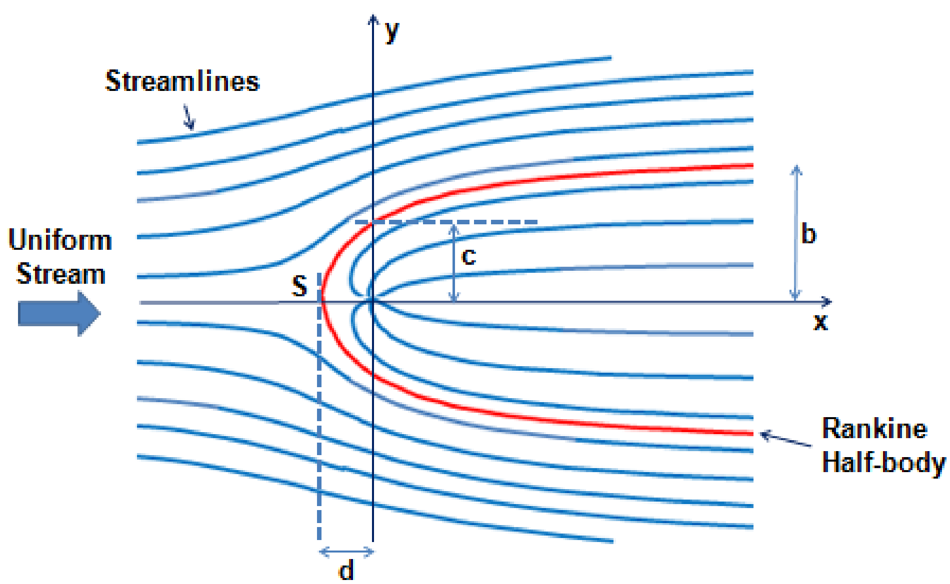

Figure 1: Planar potential flow of a source (at the origin) in a uniform stream showing streamlines and the Rankine Half-body in red.

Figure 1 shows typical streamlines including, in red, the streamline that defines a Rankine half-body; it is the streamline that crosses the x axis (which is also a streamline) at the front stagnation point, S. First we identify the distance between the front stagnation point, S, and the origin, d, by noting that, on the negative x axis, $\theta = \pi$ and therefore the velocity, u_r, is given by

$$(u_r)_{\theta=0} = -U + \frac{Q}{2\pi r} \quad \text{(Bgde4)}$$

and therefore $(u_r)_{\theta=0}$ is zero when $r = Q/2\pi U$. Therefore $d = Q/2\pi U$. Next we note that the value of the streamfunction on the negative x axis is $\psi = Q/2$ and this must also be the value of the streamfunction on the Rankine half-body streamline surface. Therefore shape of the Rankine half-body is given by the equation

$$(\psi)_{\text{Rankine halfbody}} = \frac{Q}{2} = Ur\sin\theta + \frac{Q\theta}{2\pi} \tag{Bgde5}$$

On the y axis, $\theta = \pi/2$, this yields $y = Q/4U$ so the distance c indicated in Figure 1 is $c = Q/4U$. Finally we might ask for the half-width, b, of the Rankine halfbody far downstream as $x \rightarrow \infty$. Equation (Bgde5) demonstrates that as $\theta \rightarrow 0$ the y coordinate of the half-body must asymptote to $Q/2U$ and therefore b, the half-width of the body far downstream, must be $Q/2U$. Clearly this is in accord with volume flow rate from the source acquiring the uniform stream velocity far downstream. From these geometric evaluations it is clear that there is a family of shapes of Rankine half-bodies that become increasingly streamlined as the dimension Q/U becomes smaller.