Method of Separation of Variables in Polar Coordinates

Here we will establish the form of the solutions to Laplace's equation

$$\nabla^2\phi \;=\; 0 \quad \text{or} \quad \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \quad \text{or} \quad \frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}=0 \tag{Bgda1}$$

for planar flow that result from using the method of the separation of variables in polar coordinates, r and θ, in the xy plane of the flow so that $x = r\cos\theta$ and $y = r\sin\theta$. We seek a separable solution of the form

$$\phi \;=\; R(r,t)\;\Theta(\theta,t) \tag{Bgda2}$$

where the functions $R(r,t)$ and $\Theta(\theta,t)$ need to be determined. Substituting this into equation (Bgda1) and rearranging yields

$$\frac{r^2}{R}\left\{\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}\right\} \;=\; -\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} \tag{Bgda3}$$

and since the left hand side is a function only of r and t and the right hand side is a function only of θ and t both sides can only be a function of t. Here we choose to set them both equal to a positive constant, k^2, so that

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr} \;=\; \frac{k^2R}{r^2} \quad \text{and} \quad \frac{d^2\Theta}{d\theta^2} \;=\; -k^2\Theta \tag{Bgda4}$$

and these two ordinary differential equations have the following solutions

$$R \;=\; C_{3k}r^k + C_{4k}r^{-k} \quad \text{and} \quad \Theta \;=\; C_{1k}\sin k\theta + C_{2k}\cos k\theta \tag{Bgda5}$$

where the quantities C_{1k}, C_{2k}, C_{3k}, and C_{4k}, may be constants or functions of time. Hence the form of the solution obtained by this methodology is

$$\phi \;=\; (C_{3k}r^k + C_{4k}r^{-k})(C_{1k}\sin k\theta + C_{2k}\cos k\theta) \tag{Bgda6}$$

and the velocities in the r and θ directions, u_r and u_θ, are

$$u_r \;=\; \frac{\partial\phi}{\partial r} \;=\; k(C_{3k}r^{k-1} - C_{4k}r^{-(k+1)})(C_{1k}\sin k\theta + C_{2k}\cos k\theta) \tag{Bgda7}$$

$$u_\theta \;=\; \frac{1}{r}\frac{\partial\phi}{\partial\theta} \;=\; k(C_{3k}r^{k-1} + C_{4k}r^{-(k+1)})(C_{1k}\cos k\theta - C_{2k}\sin k\theta) \tag{Bgda8}$$

In addition there is a particular solution in the special case in which $k = 0$ where the differential equations (Bgda4) become

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr} \;=\; 0 \quad \text{and} \quad \frac{d^2\Theta}{d\theta^2} \;=\; 0 \tag{Bgda9}$$

and these have the solutions

$$R \;=\; C_{30} + C_{40}\ln r \quad \text{and} \quad \Theta \;=\; C_{10} + C_{20}\theta \tag{Bgda10}$$

so that

$$\phi \;=\; (C_{30} + C_{40}\ln r)(C_{10} + C_{20}\theta) \tag{Bgda11}$$

with velocities

$$u_r \;=\; \frac{\partial\phi}{\partial r} \;=\; \frac{C_{40}}{r}(C_{10} + C_{20}\theta) \tag{Bgda12}$$

$$u_\theta \;=\; \frac{1}{r}\frac{\partial\phi}{\partial\theta} \;=\; (C_{30} + C_{40}\ln r)\frac{C_{20}}{r} \tag{Bgda13}$$