Oseen Flow

In an effort to extend the Stokes flow solutions around finite objects to finite Reynolds numbers, Oseen (1910) suggested modifying the inertial terms in the Navier-Stokes equations for steady incompressible flow,

$$\rho u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{Blh1}$$

(omitting the external force terms for convenience) by replacing the difficult non-linear inertial terms, $\rho u_j(\partial u_i/\partial x_j)$, with a linearized approximation, $\rho U(\partial u_i/\partial x_1)$ where U is the velocity in the uniform stream (in the x_1 direction) far from the object. The resulting equation of motion for Oseen flow is

$$\rho U \frac{\partial u_i}{\partial x_1} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{Blh2}$$

Though this linearization of the leading inertial term makes it possible to find an analytical solution to the flow as Oseen demonstrated, the procedure glosses over an important challenge. As can be observed by comparing the order of magnitude of the viscous and inertial terms in the Navier-Stokes equations, in a frame of reference fixed in the object the magnitude of the inertial terms asymptotes to zero at the surface since there the zero normal velocity condition and the no-slip condition require zero velocity. However the viscous terms remain finite there. On the other hand far from the surface as the flow tends toward a uniform stream the viscous terms tend to zero faster than the inertial terms and so the inertial terms come to dominate at some distance from the body. Therefore neither the Stokes equations nor the equations for potential flow (or the Oseen equations) are uniformly valid throughout the fluid domain. This is known as the *Whitehead paradox* and requires the use of matched asymptotic expansions to get around it. Clearly this is a reasonable approximation far from the body where the flow is close to being the uniform stream but is only a crude approximation close to the object. But the linearization makes it possible to seek an analytical solution to the flow.

We begin by detailing the solution for the flow around a sphere that Oseen uncovered using the equations (Blh2) with the linearized inertial terms. By satisfying both the zero normal velocity condition and the no-slip condition on the surface of the sphere, $r = R$, he obtained a solution in spherical coordinates, (r, θ, ϕ) that is also detailed, for example, in Yih (1969). The Stokes streamfunction, $\psi(r, \theta, \phi)$, of that solution is

$$\frac{\psi}{UR^2} = \frac{(R^3 + 2r^3)\sin^2\theta}{4R^2 r} - \frac{3}{Re}(1+\cos\theta)\left(1 - e^{-Re\; r(1-\cos\theta)/4R}\right) \tag{Blh3}$$

where $Re = 2UR/\nu$ is the conventional Reynolds number of the flow. Consequently, unlike the Stokes flow solution which is symmetric about the plane at $\theta = \pi/2$, this Oseen flow is asymmetric and that asymmetry is related to the non-zero value of the Reynolds number. As the Reynolds number increases the asymmetry increases. Expanding for small Reynolds number, Re, equation (Blh3) becomes

$$\frac{\psi}{UR^2} = \left\{\frac{R}{4r} + \frac{r^2}{2R^2} - \frac{3r}{4R}\right\}\sin^2\theta + \frac{3r^2 Re}{32R^2}(1-\cos\theta)\sin^2\theta + O(Re^2) \tag{Blh4}$$

Indeed, as seen in figures 2 and 1, the solution exhibits an asymmetry that is reminiscent of a rudimentary wake.

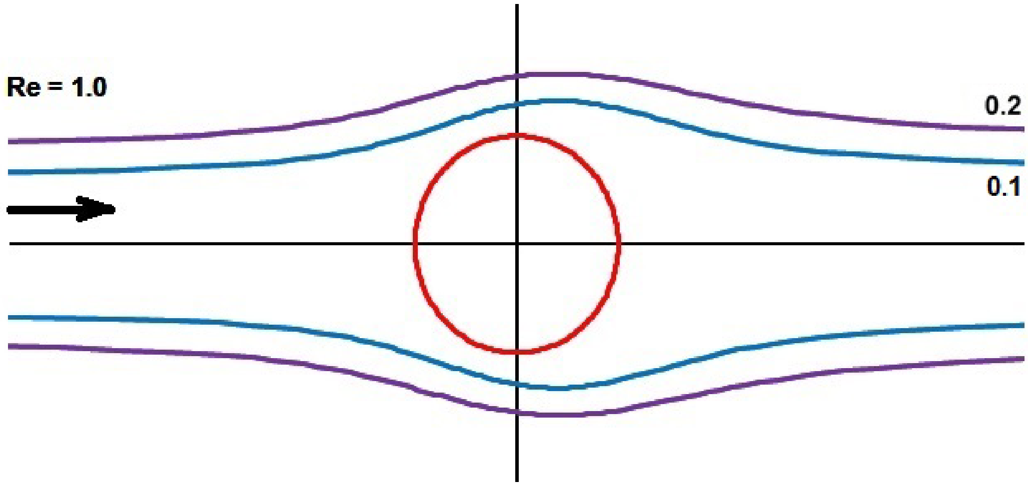

Figure 1: Typical streamlines in Oseen flow past a sphere: The streamlines for $\psi/UR^2 = 0.1$ and 0.2 at a Reynolds number of $Re = 1.0$.

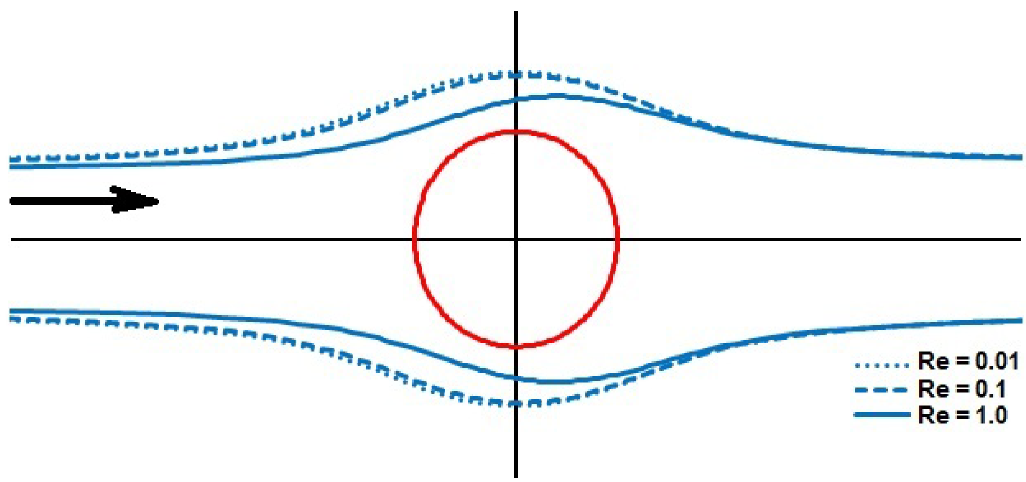

Figure 2: Typical streamlines in Oseen flow past a sphere: The streamline for $\psi/UR^2 = 0.1$ at three different Reynolds numbers of $Re = 0.01$, 0.1 and 1.0.

When the drag on the sphere in Oseen flow is evaluated it transpires that

$$\text{Drag} \;=\; 6\pi\mu RU\left[1+\frac{3Re}{16}\right] \tag{Blh4}$$

However, because the flow close to the surface is not accurately modeled by the Oseen equation (Blh2), the Reynolds number correction to the classic Stokes flow result for the drag, $6\pi\mu RU$, might be questioned though it turns out to agree with the more accurately evaluated correction term at the $O(Re)$ order. Proudman and Pearson (1957) carried out a more accurate analysis in which a Stokes flow solution near the surface of the sphere and a perturbation solution in the far field that accounted for the inertial terms were dovetailed together using the method of matched asymptotic expansions. Their expression for the drag was

$$\text{Drag} \;=\; 6\pi\mu RU\left[1+\frac{3Re}{16}+\frac{9Re^2\ln{(Re/2)}}{160}+O(Re^2)\right] \tag{Blh5}$$

and demonstrates that the $O(Re)$ correction of Oseen was correct (though perhaps fortuitously).

The above-mentioned paradox in the three-dimensional flow around finite objects becomes even more serious when it comes to planar flows. It transpires that the mis-match caused by the dominance of the viscous terms close to the surface of the body and the dominance of the inertial terms far away mean that it is not possible to find a uniformly valid solution to the external, low Reynolds number viscous flow of a uniform stream around a finite body. This is known as *Stokes paradox*.

We note that linearization of these far-field inertia effects by means of Oseens approximation permits the construction of more complicated flow fields by means of a modified set of fundamental solutions (see section (Blc)) in which the *Oseenlet* replaces the Stokeslet. Then more complicated flows at low Reynolds number that include approximate inertial effects can be constructed by superposition using Oseenlets and other singularities.