Turbulent Pipe Flows

One example of the application of the universal turbulent velocity profile is to fully-developed turbulent flow in a circular pipe. It transpires that the application of the profile all the way from the interior surface of the pipe ($r = R$ or $y = R - r = 0$ where r is the radial position in the axisymmetric pipe and R is its interior radius) to the central axis ($r = 0$) proves an adequate model of the flow and its consequences. We begin with the case of a smooth interior pipe surface and use the following universal velocity profile from section (Bkj):

$$u^* = y^* \qquad \text{for} \qquad 0 < y < \delta_{LSL} \tag{Bkl1}$$

$$u^* \;=\; \frac{1}{\kappa}\ln y^* + C \;=\; \frac{2.303}{\kappa}\, log_{10}(y^*) + C \qquad \text{for} \qquad \delta_{LSL} < y < R \tag{Bkl2}$$

where, as previously defined in section (Bkj), $u^* = \overline{u}/u_\tau$ and $y^* = u_\tau y/\nu$.

The first step is to find the relation between the volume flow rate in the pipe, Q, or, equivalently, the volumetric average velocity, V, given by

$$V \;=\; \frac{Q}{\pi R^2} \;=\; \frac{2}{R^2}\int_0^R \overline{u}r\; dr \tag{Bkl3}$$

and the wall shear stress, τ_W. The wall shear stress is, in turn, related to the longitudinal pressure gradient, dp/dx, by

$$\tau_W \;=\; -\frac{R}{2}\left(\frac{dp}{dx}\right) \tag{Bkl4}$$

through a basic application of the momentum thereom. Similarly, the shear stress, σ_{xy}, at any radial position, r is related to dp/dx by

$$\sigma_{xy} \;=\; -\frac{r}{2}\left(\frac{dp}{dx}\right) \tag{Bkl5}$$

Before proceeding with the derivation, we should also note the following general definitions pertaining to pressure gradient and wall shear stress in pipe flow. First the *friction factor*, f, so widely used in practical engineering calculations is defined by

$$f \;=\; \frac{4R}{\rho V^2}\left(-\frac{dp}{dx}\right) \;=\; \frac{8u_\tau^2}{V^2} \;=\; \frac{8\tau_w}{\rho V^2} \tag{Bkl6}$$

and we can anticipate that the Reynolds number of flow, $Re = 2RV/\nu$, will appear in the dimensionless results of this calculation.

With these definitions in mind we can now substitute a sample universal velocity profile, in particular,

$$\overline{u} \;=\; u_\tau\left[5.75\; log_{10}(y^*) + 5.5\right] \tag{Bkl7}$$

into the expression (Bkl13) for V and, after some manipulation write the answer as

$$\frac{V}{2u_\tau} \;=\; \left(\frac{2}{f}\right)^{\frac{1}{2}} \;=\; \frac{2}{R^2}\int_0^R u_\tau(R-y)\left[5.75\; log_{10}(\frac{yu_\tau}{\nu}) + 5.5\right] dy \tag{Bkl8}$$

Completing the integration the result can be written as

$$\frac{1}{f^{\frac{1}{2}}} \;=\; 2\ log_{10}\left\{f^{\frac{1}{2}}Re\right\} - 0.9 \tag{Bkl9}$$

where the last constant varies depending on the data used to obtain it. This implicit relationship between the friction factor, f, and the Reynolds number, Re, is readily computed. The results are traditionally presented in the *Moody diagram*, a graph of the friction factor against the Reynolds number, included in Figure 1. The relation (Bkl9) is represented by the line labelled turbulent flow in smooth pipes; the result

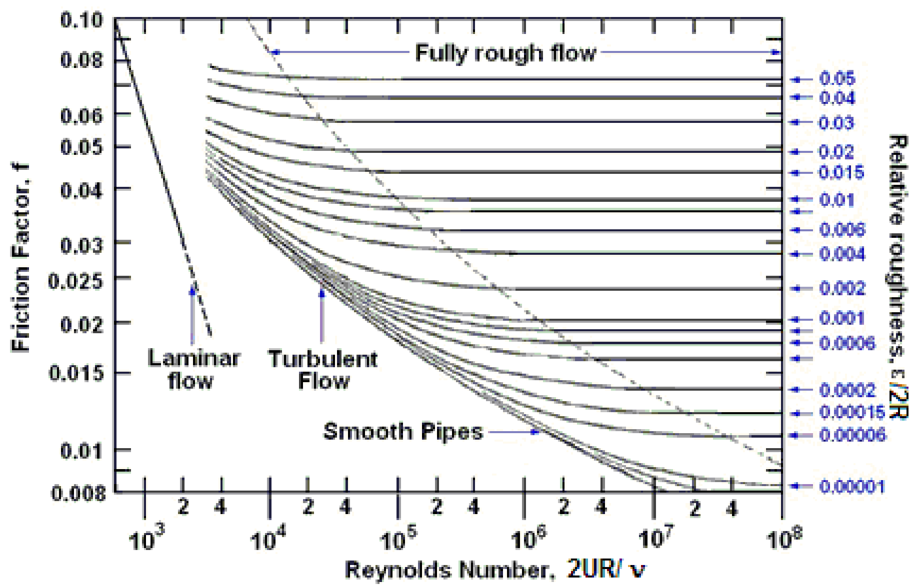

Figure 1: The Moody diagram, the friction factor, f, for flow in a circular pipe plotted against the Reynolds number, $2VR/\nu$.

for laminar flow, namely $f = 64/Re$, is included on the left side of the graph and the lines for rough pipes will be addressed shortly.

The velocity profile in the smooth pipe follows from the universal velocity profile at a given Reynolds number, Re, when the value of f is first obtained by solution of equation (Bkl9), and that expression is substituted back into equation (Bkl7) to obtained the result that

$$\frac{\overline{u}}{V} \;=\; 1.0 + 1.33f^{\frac{1}{2}} + 2f^{\frac{1}{2}}\ log_{10}(\frac{y}{R}) \tag{Bkl10}$$

which also implies that

$$\frac{\overline{u_{r=0}}}{V} \;=\; 1.0 + 1.33f^{\frac{1}{2}} \tag{Bkl11}$$

and

$$\frac{\overline{u}}{\overline{u_{r=0}}} \;=\; 1.0 + \frac{2f^{\frac{1}{2}}\ log_{10}(\frac{y}{R})}{1.0 + 1.33f^{\frac{1}{2}}} \tag{Bkl12}$$

This implies blunter velocity profiles as the Reynolds number is increased and this is exemplified by the sample velocity profiles shown in Figure 3.

When the cruder Blasius velocity profile is used instead of the universal velocity profile the corresponding result is

$$f \;=\; \frac{0.309}{Re^{1/4}} \tag{Bkl13}$$

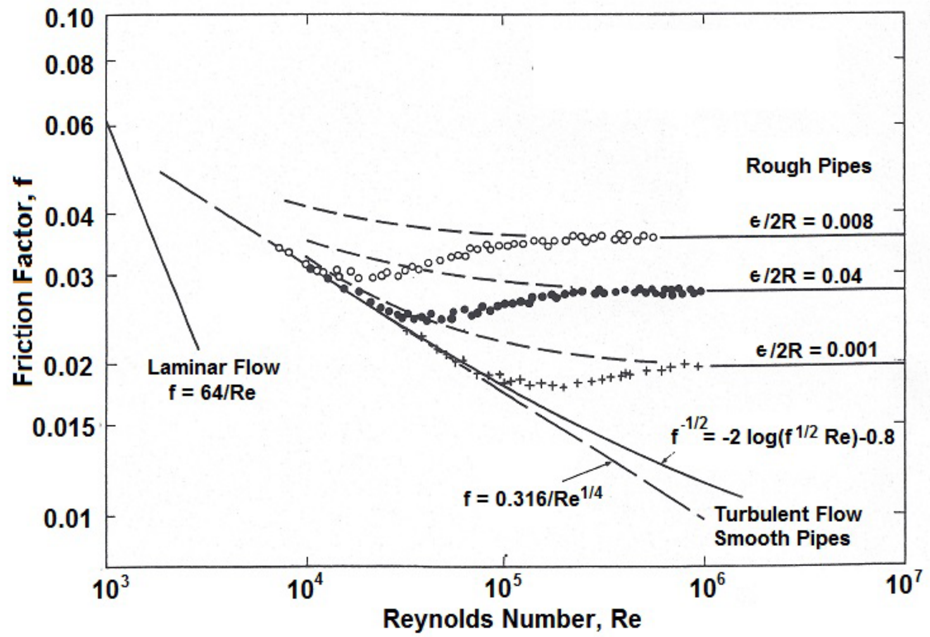

Figure 2: Moody diagram of friction factor, f, for a circular pipe plotted against Reynolds number, $2RV/\nu$.

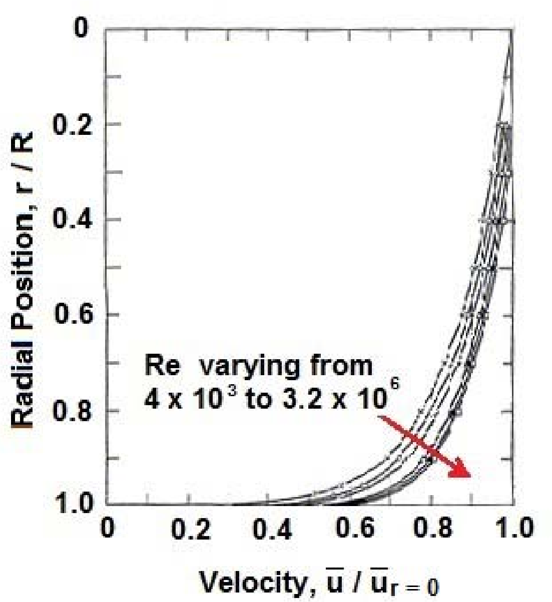

Figure 3: Changes in the velocity profile of the turbulent flow in a smooth pipe as the Reynolds number, Re, changes from 4.0×10^3 to 3.2×10^6.

As can be seen in Figure 2 the result for this velocity profile is very close to that from equation (Bkl9) except at the larger Reynolds numbers. The corresponding velocity profile is

$$\frac{\overline{u}}{\overline{u_{r=0}}} = \left(\frac{y}{R}\right)^{\frac{1}{7}} \tag{Bkl14}$$