Wave Propagation in Ducts

In order to solve unsteady flows in ducts, an expression for the sonic speed is combined with the differential form of the equation for conservation of mass (the continuity equation),

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial s}(\rho A u) = 0 \tag{Bnfb1}$$

where $u(s,t)$ is the cross-sectionally averaged or volumetric velocity, s is a coordinate measured along the duct, and t is time. The appropriate differential form of the momentum equation is

$$\rho \left[\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial s} \right] = -\frac{\partial p}{\partial s} - \rho g_s - \frac{\rho f u |u|}{4a} \tag{Bnfb2}$$

where g_s is the component of the acceleration due to gravity in the s direction, f is the friction factor, and a is the radius of the duct.

Now the barotropic assumption (Bnfa3) allows the terms in equation (Bnfb1) to be written as

$$\frac{\partial}{\partial t}(\rho A) = \frac{A}{c^2}\frac{\partial p}{\partial t} \quad ; \quad \frac{\partial(\rho A)}{\partial s} = \frac{A}{c^2}\frac{\partial p}{\partial s} + \rho \frac{\partial A}{\partial s}\Big|_p \tag{Bnfb3}$$

so the continuity equation becomes

$$\frac{1}{c^2}\frac{\partial p}{\partial t} + \frac{u}{c^2}\frac{\partial p}{\partial s} + \rho \left[\frac{\partial u}{\partial s} + \frac{u}{A}\frac{\partial A}{\partial s}\Big|_p \right] = 0 \tag{Bnfb4}$$

Equations (Bnfb2) and (Bnfb4) are two simultaneous, first order, differential equations for the two unknown functions, $p(s,t)$ and $u(s,t)$. They can be solved given the barotropic relation for the fluid, $\rho(p)$, the friction factor, f, the normal cross-sectional area of the pipe, $A_0(s)$, and boundary conditions which will be discussed later. Normally the last term in equation (Bnfb4) can be approximated by $\rho u(dA_0/ds)/A_0$. Note that c may be a function of s.

In the time domain methodology, equations (Bnfb2) and (Bnfb4) are normally solved using the method of characteristics (see, for example, Abbott 1966). This involves finding moving coordinate systems in which the equations may be written as ordinary rather than partial differential equations. Consider the relation that results when we multiply equation (Bnfb4) by λ and add it to equation (Bnfb2):

$$\rho \left[\frac{\partial u}{\partial t} + (u+\lambda)\frac{\partial u}{\partial s} \right] + \frac{\lambda}{c^2}\left[\frac{\partial p}{\partial t} + \left(u + \frac{c^2}{\lambda} \right) \frac{\partial p}{\partial s} \right]$$

$$+ \frac{\rho u \lambda}{A_0}\frac{dA_0}{ds} + \rho g_s + \frac{\rho f |u| u}{4a} = 0 \tag{Bnfb5}$$

If the coefficients of $\frac{\partial u}{\partial s}$ and $\frac{\partial p}{\partial s}$ inside the square brackets were identical, in other words if $\lambda = \pm c$, then the expressions in the square brackets could be written as

$$\frac{\partial u}{\partial t} + (u \pm c)\frac{\partial u}{\partial s} \quad \text{and} \quad \frac{\partial p}{\partial t} + (u \pm c)\frac{\partial p}{\partial s} \tag{Bnfb6}$$

and these are the derivatives $\frac{du}{dt}$ and $\frac{dp}{dt}$ on $\frac{ds}{dt} = u \pm c$. These lines $\frac{ds}{dt} = u \pm c$ are the characteristics, and on them we may write:

1. In a frame of reference moving with velocity $u+c$ or on $\frac{ds}{dt}=u+c$:

$$\frac{du}{dt}+\frac{1}{\rho c}\frac{dp}{dt}+\frac{uc}{A_0}\frac{dA_0}{ds}+g_s+\frac{fu|u|}{4a}=0 \tag{Bnfb7}$$

2. In a frame of reference moving with velocity $u-c$ or on $\frac{ds}{dt}=u-c$:

$$\frac{du}{dt}-\frac{1}{\rho c}\frac{dp}{dt}-\frac{uc}{A_0}\frac{dA_0}{ds}+g_s+\frac{fu|u|}{4a}=0 \tag{Bnfb8}$$

A simpler set of equations result if the piezometric head, h^*, defined as

$$h^*=\frac{p}{\rho g}+\int\frac{g_s}{g}ds \tag{Bnfb9}$$

is used instead of the pressure, p, in equations (Bnfb7) and (Bnfb8). In almost all hydraulic problems of practical interest $p/\rho_L c^2 \ll 1$ and, therefore, the term $\rho^{-1}dp/dt$ in equations (Bnfb7) and (Bnfb8) may be approximated by $d(p/\rho)/dt$. It follows that on the two characteristics

$$\frac{1}{\rho c}\frac{dp}{dt}\pm gs \quad\approx\quad \frac{g}{c}\frac{dh^*}{dt}-\frac{u}{c}g_s \tag{Bnfb10}$$

and equations (Bnfb7) and (Bnfb8) become

1. On $\frac{ds}{dt}=u+c$

$$\frac{du}{dt}+\frac{g}{c}\frac{dh^*}{dt}+uc\frac{1}{A_0}\frac{dA_0}{ds}-\frac{ug_s}{c}+\frac{f}{4a}u|u|=0 \tag{Bnfb11}$$

2. On $\frac{ds}{dt}=u-c$

$$\frac{du}{dt}-\frac{g}{c}\frac{dh^*}{dt}-uc\frac{1}{A_0}\frac{dA_0}{ds}+\frac{ug_s}{c}+\frac{f}{4a}u|u|=0 \tag{Bnfb12}$$

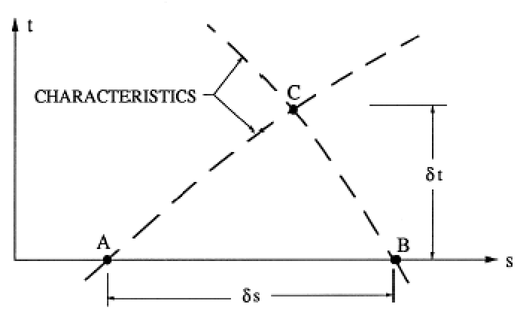

Figure 1: Method of characteristics.

These are the forms of the equations conventionally used in unsteady hydraulic water-hammer problems (Streeter and Wylie 1967). They are typically solved by relating the values at a time $t+\delta t$ (for example point C of figure 1) to known values at the points A and B at time t. The lines AC and BC are characteristics, so the following finite difference forms of equations (Bnfb11) and (Bnfb12) apply:

$$\frac{(u_C-u_A)}{\delta t}+\frac{g}{c_A}\frac{(h^*_C-h^*_A)}{\delta t}+u_Ac_A\left(\frac{1}{A_o}\frac{dA_0}{ds}\right)_A-\frac{u_A(g_s)_A}{c_A}+\frac{f_Au_A|u_A|}{4a}=0 \tag{Bnfb13}$$

and

$$\frac{(u_C - u_B)}{\delta t} - \frac{g}{c_B}\frac{(h_C^* - h_B^*)}{\delta t} - u_B c_B \left(\frac{1}{A_o}\frac{dA_0}{ds}\right)_B + \frac{u_B (g_s)_B}{c_B} + \frac{f_B u_B |u_B|}{4a} = 0 \tag{Bnfb14}$$

If $c_A = c_B = c$, and the pipe is uniform, so that $dA_0/ds = 0$ and $f_A = f_B = f$, then these reduce to the following expressions for u_C and h_C^*:

$$u_C = \frac{(u_A + u_B)}{2} + \frac{g}{2c}(h_A^* - h_B^*) + \frac{\delta t}{2c}\left[u_A (g_s)_A - u_B (g_s)_B\right] - \frac{f\delta t}{8a}\left[u_A|u_A| + u_B|u_B|\right] \tag{Bnfb15}$$

$$h_C^* = \frac{(h_A^* + h_B^*)}{2} + \frac{c}{2g}(u_A - u_B) + \frac{\delta t}{2g}\left[u_A (g_s)_A + u_B (g_s)_B\right] - \frac{fc\delta t}{8ag}\left[u_A|u_A| - u_B|u_B|\right] \tag{Bnfb16}$$