General Relation between Lift and Circulation

Having, in the previous section (Dcb), derived the relation between the circulation, Γ, on a circular cylinder spinning in a uniform stream and the lift, L, experienced by the cylinder we shall now proceed to prove that the same result holds for **any** two-dimensional object in a steady, planar, potential flow.

The concept of circulation is very important to our understanding of the lift on any object in a fluid flow. This understanding begins with Kelvin's theorem which was proven in an earlier section (section (Bdj)) and states that in the flow of an inviscid fluid under the action of conservative body forces, the circulation around any closed contour within the fluid will not change with time. Now consider the inviscid flow (with conservative body forces) of a uniform stream of velocity, U, around an object such as the airfoil depicted in Figure 1. In this flow consider a general contour which at some much earlier moment was far upstream of the object. Being entirely in the uniform stream any contour would therefore have zero circulation. Now, consider it becomes wrapped around the object as shown in Figure 1. Since its circulation did not change with time it must still have zero circulation. But now consider such a contour comprised of four

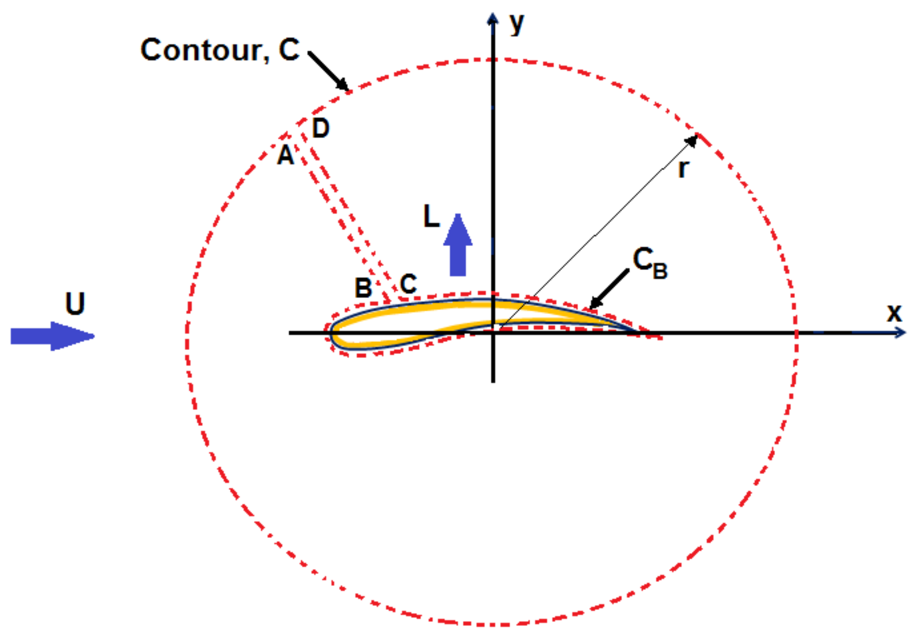

Figure 1: Closed contours around an airfoil.

parts, namely

1. The contour, C, that surrounds the object except for an infinitesmal interruption between the locations A and D.

2. Two essentially colocated sections AB and CD on either side of a cut connecting the contours C and C_B.

3. The contour, C_B, which surrounds the object except for an infinitesmal interruption between the locations B and C.

It follows that we can decompose the full contour integral whose circulation is zero into four line parts namely

$$\oint \equiv \int_C + \int_{AB} + \int_{CD} + \int_{C_B} = 0 \tag{Dcb1}$$

But the integrals along AB and CD are not only colocated but also in opposite directions and so the sum of those two integrals must be zero so that the relation (Dcb1) becomes

$$\oint \equiv \int_C + \int_{C_B} = 0 \tag{Dcb2}$$

Now the direction of evaluation of the remaining two integrals are opposite, the integral along C being in the anticlockwise direction whereas the integral along C_B is in the clockwise direction. It therefore follows from equation (Dcb2) that

$$\oint_C = \oint_{C_B} \tag{Dcb3}$$

where both closed contour integrals are defined as positive in the anticlockwise direction as was established as the convention in section (Bdj). It follows that the circulation evaluated around the contour on the surface of the object or any other contour surrounding the object is the same; we denote both by Γ.

The importance of this result lies in the fact that the circulation, Γ, around any object such as an airfoil can be evaluated using any contour we wish that surrounds the object. In section (Bdgj) we demonstrated that application of the momentum theorem to such a flow configuration led to the relation that the lift on the object, L, in the y direction is directly related to the circulation Γ by

$$L = -\rho U \Gamma \tag{Dcb4}$$

Of course, as yet, it is unclear what determines the circulation, Γ, and this too requires further discussion including analyses of the effects of viscosity. We earlier developed a special case of this result, namely the lift on a spinning cylinder in potential flow but the astute reader will have recognized that we did not explain how the circulation was transmitted to the fluid by the spinning cylinder. We only observed that if such circulation existed then the lift on the cylinder (known in that case as the Magnus Force) would be given by $L = -\rho\Gamma U$. However, it is an effect well known in practice through the effects of spin on the flight of golf balls, baseballs and soccer balls.

Finally we should take note of the fact that throughout this and the preceding sections like section (Bgdj) the circulation has been defined as positive in the anticlockwise direction (as a result of the polar velocity, u_θ, being defined as positive in the anticlockwise direction) and this determines the negative sign in equation (Dcb4). However, it should be noted that the opposite sign convention is often employed for the circulation in discussion of airfoil fluid dynamics and this leads to $L = \rho U \Gamma$; thus, in individual analyses, care needs to be exercised to define the sign convention being used.