Two-dimensional Flow Analysis

It is useful at this point to develop an approximate and idealized evaluation of the hydraulic performance of a turbine in the absence of cavitation. This will take the form of an analytical expression for the head drop (or ψ) as a function of the flow rate (or ϕ). To simplify this analysis it is assumed that the flow through the runner is incompressible, axisymmetric and steady in the rotating framework of the runner blades; that the blades are infinitely thin. Viscous losses will momentarily be neglected. Under these

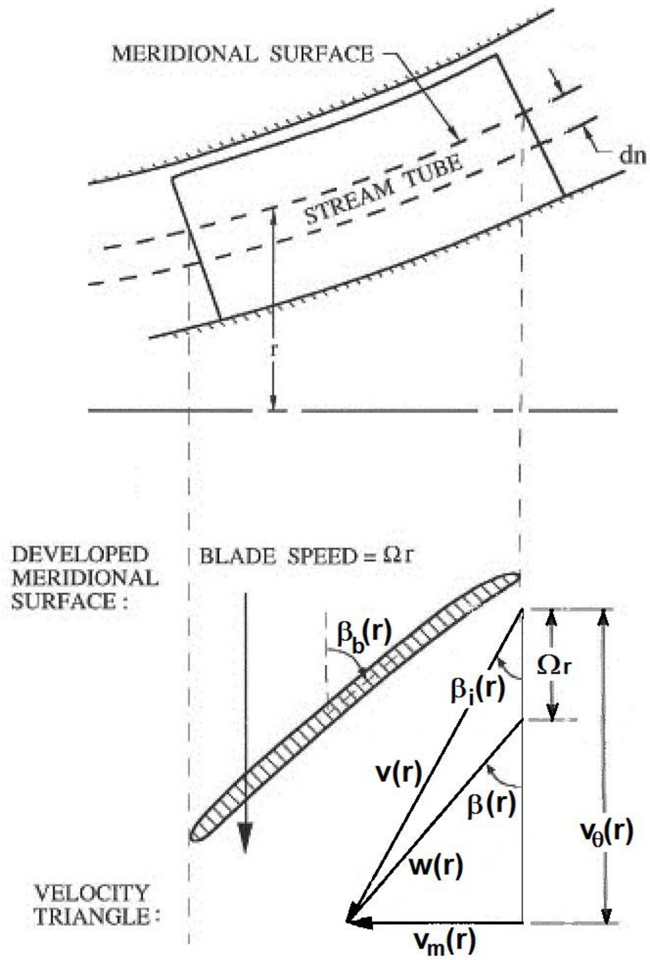

Figure 1: Developed meridional surface and velocity triangle.

conditions the flow in any streamtube, such as depicted in Figure 1), will follow the Bernoulli equation for a rotating system (see, for example, Sabersky *et al.* 1989),

$$\frac{2p_1}{\rho} + w_1^2 - r_1^2\Omega^2 = \frac{2p_2}{\rho} + w_2^2 - r_2^2\Omega^2 \tag{Mdi1}$$

This equation can be usefully interpreted as an energy equation as follows. The terms $p + \frac{1}{2}\rho w^2$ on either side are the total pressure or mechanical energy per unit volume of fluid, and this quantity would be the same at inlet and discharge were it not for the fact that "potential" energy is stored in the rotating fluid. The term $\rho(r_1^2 - r_2^2)\Omega^2/2$ represents the difference in this "potential" energy at inlet and discharge.

Clearly, when there are losses, equation (Mdi1) will no longer hold exactly but will be empirically modified to account for viscous losses.

Using the definition of the total pressure (equation (Mde6)) and the relations between the velocities derived from the velocity triangles of figure 1, equation (Mdi1) can be manipulated to yield the following expression for the total pressure drop, $(p_1^T - p_2^T)$, for a given streamtube:

$$p_1^T - p_2^T = p_1 - p_2 + \frac{\rho}{2}\left(v_1^2 - v_2^2\right) \tag{Mdi2}$$

and, if the hydraulic losses are included, the relation (Mdi2) is traditionally and empirically modified using a turbine hydraulic efficiency, η_T, so that

$$\frac{\eta_T}{\rho\Omega}\left\{p_1^T - p_2^T\right\} \quad = \quad r_1 v_{\theta 1} - r_2 v_{\theta 2} \quad = \quad r_1(v_{m1}\cot\beta_1 + \Omega r_1) - r_2(v_{m2}\cot\beta_2 + \Omega r_2) \tag{Mdi3}$$

using geometric relations from the velocity triangles of Figure 1.

This expression for the head drop in an individual stream tube can then used to integrated over the entire flow through the runner using the expressions (Mde12) and (Mde13) (as was done in the equivalent hydraulic analysis for a pump described in section (Mbdc)). A more approximate but still useful result can be achieved by utilizing the tip geometry in equation (Mdi3) by substituting

$$r_1 \approx R_{T1} \quad ; \quad r_2 \approx R_{T2} \quad ; \quad \beta_1 \approx \beta_{T1} \quad ; \quad \beta_2 \approx \beta_{T2} \quad ; \quad v_{m1} = Q/A_1 \quad ; \quad v_{m2} = Q/A_2 \tag{Mdi4}$$

and, from equation (Mde1):

$$\cot\beta_{T1} = \cot\beta_i - \frac{A_1}{R_{T1}^2\phi} \tag{Mdi5}$$

where A_1 is the cross-sectional area of the flow at inlet (and A_2 is the cross-sectional area of the flow at discharge from the runner). Therefore, using the definitions of equations (Mde7) and (Mde8), equation (Mdi3) can be rearranged to express the following form of the non-dimensional performance characteristic of the turbine:

$$\eta_T\ \psi \ = \ \phi\left\{\frac{R_{T1}^2}{A_1}\right\}\left\{\cot\beta_{T1} - \frac{R_{T2}}{R_{T1}}\frac{A_1}{A_2}\cot\beta_{T2}\right\} + \left\{1 - \frac{R_{T2}^2}{R_{T1}^2}\right\} \tag{Mdi6}$$

using the definitions in equations (Mde7). Alternatively utilizing the relation (Mdi5):

$$\eta_T\ \psi \ = \ \phi\left\{\frac{R_{T1}^2}{A_1}\right\}\left\{\cot\beta_i - \frac{R_{T2}}{R_{T1}}\frac{A_1}{A_2}\cot\beta_{T2}\right\} - \frac{R_{T2}^2}{R_{T1}^2} \tag{Mdi7}$$

For the convenience of later discussion we write this as

$$\eta_T\ \psi \ = \ \Gamma_1\phi - \Gamma_2 \tag{Mdi8}$$

where

$$\Gamma_1 \ = \ \left\{\frac{R_{T1}^2}{A_1}\right\}\left\{\cot\beta_i - \frac{R_{T2}}{R_{T1}}\frac{A_1}{A_2}\cot\beta_{T2}\right\} \quad \text{and} \quad \Gamma_2 \ = \ \frac{R_{T2}^2}{R_{T1}^2} \tag{Mdi9}$$

Clearly an approximate, two-dimensional analysis and an expression like (Mdi7) will be more accurate for cases in which the flows at inlet and discharge are uniform as would be the case for a turbine in which the widths, B_1 and B_2 (figure 1 of section (Mde)), are such that $B_1 \ll R_{T1}, B_2 \ll R_{T2}$, and in which the velocities of the flow are uniform across both the inlet and the discharge. However, even in nonuniform cases in which the two-dimensional analysis is less appropriate and in which it is more appropriate to subdivide the flow into stream tubes and integrate over those stream tubes using equations (Mde12) and

(Mde13), a linear relation like (Mdi8) is still a useful approximation and the simple expressions (Mdi9) can be used in combination with mean or effective angles, β_i and β_{T2}, and geometric ratios, R_{T2}/R_{T1} and A_1/A_2, to estimate the performance of the turbine. Moreover, as in the case of the equivalent pump analysis, various empirical loss coefficients can be added to this analysis in order to evaluate the efficiency, η_T, and thereby achieve fairly close agreement with experimental measurements. We also note is passing that, from equation (Mde18), the torque/power coefficient, $\mathcal{T}$, is related to ψ and ϕ by $\mathcal{T} = \phi\psi$.

For convenience we focus our comments on the non-dimensional performance given by equation (Mdi7) or (Mdi8). First note that, just as was the case for the equivalent analysis of a pump, the ideal or lossless ($\eta_T = 1$) non-dimensional head coefficient, $\eta_T\psi$, varies linearly with the non-dimensional flow coefficient, ϕ. In a hydraulic performance graph of $\eta_T\psi$ plotted versus ϕ the intercept with the vertical or $\eta_T\psi$ axis should be approximately given by $-\Gamma_2$ and thus be only a function of the ratio of the discharge radius to the inlet radius. However, the slope of the performance curve will vary with the wicket gate setting which sets the inlet swirl angle, β_i, in the non-rotating frame and with the discharge blade angle, β_{T2}, in the rotating frame. An example is presented in the next section.