Fluid Statics

In order to determine how the pressure varies from point to point within a fluid at rest, consider a small circular cylinder (cross-sectional area, A, and height, dy) of fluid (density ρ) with its axis oriented vertically. As usual we employ a coordinate framework in which y is oriented vertically upward and the coordinates x and z are in a horizontal plane. The acceleration due to gravity is denoted by g.

We will evaluate all the forces acting vertically on this cylinder. First the weight of the fluid within the cylinder, $\rho g A dy$, acts vertically downward. Since the fluid is at rest there will be no shear forces acting vertically on the sides of the cylinder. If we define the pressure acting on the bottom surface as p then the force acting vertically upward on that surface is pA. Since the pressure on the top surface will therefore be $p + (\partial p/\partial y)dy$, it follows that the force acting downward on that upper surface is $\{p + (\partial p/\partial y)dy\}\, A$.

Since the fluid cylinder is at rest all these vertical forces must sum to zero and hence it follows that

$$\frac{\partial p}{\partial y} = -\rho g \tag{Cb1}$$

Thus the vertical gradient of the pressure is equal to the density multiplied by the acceleration due to gravity. We will pursue the consequences of this result shortly.

Next we will evaluate all the forces acting horizontally on a similar cylinder with its axis oriented in a horizontal direction. The weight of the fluid within the cylinder does not contribute to the horizontal forces; moreover, there will be no shear forces acting on the sides or ends of the cylinder since the fluid is at rest. Consequently, the forces due to the pressures on the ends of the cylinder must balance and therefore the pressure gradient in any horizontal direction must be zero or

$$\frac{\partial p}{\partial x} = 0 \qquad ; \qquad \frac{\partial p}{\partial z} = 0 \tag{Cb2}$$

In other words in a connected body of fluid at rest, the pressure must be the same everywhere on any horizontal plane. Note that since the pressure only varies with y the partial derivative, $\partial p/\partial y$ in the above equation may be replaced by dp/dy.

The above results completely define the distribution of pressure within a connected body of fluid at rest. However, it remains to integrate the first result in order to explicitly generate an expression for the pressure, p, in terms of the vertical coordinate or elevation, y:

$$p = p_0 - \int_{y_0}^{y} \rho g dy \tag{Cb3}$$

where p_0 is the reference pressure at some reference location, $y = y_0$. If the density were the same everywhere and the acceleration due to gravity were the same everywhere then ρg would be a simple constant and the integration leads to

$$p = p_0 - \rho g (y - y_0) \tag{Cb4}$$

This simple result is applicable in many situations where ρg is close to being constant, for example in almost all **terrestrial tanks of liquid**, **manometers** or in **lakes and oceans**. It also leads directly to **Archimedes principle**. Moreover, the further integration of the pressure over a surface in contact with the fluid leads to important results for the **forces experienced by walls or submerged objects**.

In **planetary atmospheres** on the other hand the density (and to some extent g) varies significantly with altitude and the integration is more complex.

The **interiors of planetary or celestial objects** require a different integration since, even when the density is relatively constant, the acceleration due to gravity, g, will vary significantly with radius.