Measurements using Bernoulli's equation

Many fluid measurement devices and techniques are based on Bernoulli's equation and we list them here with analysis and discussion.

Venturi Meter

Venturi meters are widely used to measure the volume flow rate in piping system. They consist of a Venturi nozzle, a smooth restriction designed to minimize the hydraulic losses introduced by this added restriction in the flow path. Static pressure taps are installed upstream of the restriction and at the

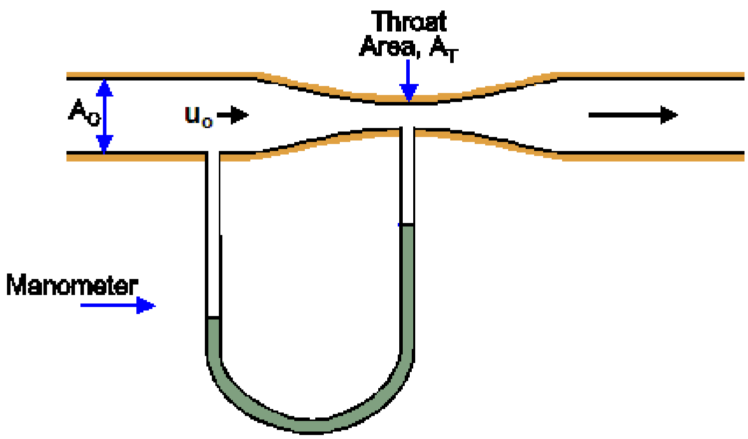

Figure 1: Venturi Flow Meter.

point of minimum cross-sectional area or throat of the device to measure the static pressures at those locations, denoted respectively by p_o and p_T. Then, if the crossectional areas upstream and at the throat are respectively denoted by A_o and A_T and the viscous losses between these locations are neglected, it follows from Bernoulli's equation that if the fluid is steady, incompressible and inviscid then

$$\frac{1}{2}u_o^2 + p_o \;=\; \frac{1}{2}u_T^2 + p_T \tag{Kdcb1}$$

(assuming the device is horizontal). Since by continuity, $u_o A_o = u_T A_T$, it follows that

$$u_o \;=\; \left[\frac{2(p_o - p_T)}{\rho\left\{\frac{A_o^2}{A_T^2} - 1\right\}}\right]^{\frac{1}{2}} \tag{Kdcb2}$$

It follows that the upstream volume-averaged velocity, u_o, and therefore the volume flow rate, $A_o u_o$, can be calculated from a measured pressure difference, $(p_o - p_T)$, provided the fluid density, ρ, and the areas, A_o and A_T, are known. If a manometer is used to measure that pressure difference then the quantity, $(p_o - p_T)/\rho g$, often follows directly from the difference in the elevation levels of the manometer. This makes it particularly convenient to calculate the flow velocity, u_o.

Venturi meters are most useful for devices in which the Reynolds number, $Re = u_o A_o^{\frac{1}{2}}/\nu$ is much larger than 1000 because lower Reynolds numbers lead to significant viscous losses that are unaccounted for. At the other end of the scale some Venturi meters are very large indeed. Figure 2 is a photograph of

Figure 2: Venturi flow meter in a 15ft diameter water supply line near Lake Mead, Nevada.

a Venturi meter installed in $15ft(?)$ diameter water supply conduit near Lake Mead, Nevada. Note the lengthy diffuser downstream of the throat designed to minimize the hydraulic losses in the device.

Orifice Meter

Because of the length of a Venturi meter it is not always easy or convenient to incorporate it in a hydraulic

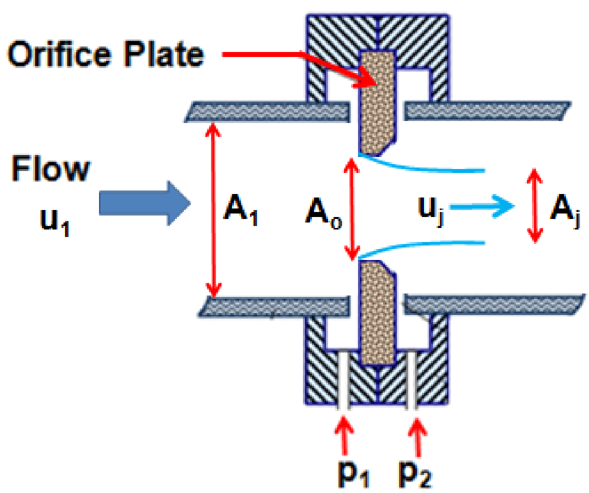

Figure 3: Orifice Flow Meter.

system; moroever, a Venturi meter can be expensive to manufacture. A much more convenient version consists of a simple, flat plate orifice installed between two flanges in the pipe system as depicted in Figure

3. The flow through the orifice forms a jet as sketched in Figure 3 and the pressure, p_2, that is measured by the downstream pressure tap adjusts to the jet velocity, U_j, according to Bernoulli's equation so that

$$\frac{1}{2}\rho u_1^2 + p_1 = \frac{1}{2}\rho u_j^2 + p_2 \quad \text{(Kdcb3)}$$

However the simplicity of manufacture of the orifice meter is somewhat offset by the fact that the jet velocity, u_j, is not immediately known because the jet cross-sectional area, A_j, is significantly less than the area of the orifice, A_o. Since $u_j = u_1 A_1/A_j$ it follows that u_1 cannot be calculated from $(p_1 - p_2)$ unless the area ratio, A_j/A_o is known. This needed quantity, A_j/A_o, is called the **contraction coefficient** and is denoted by C_c. Fortunately the contraction coefficient for a sharp-edged orifice is normally close to about 0.6 and, with that or some other known value, the volume-averaged velocity in the upstream pipe can be calculated using a slightly modified version of equation (Kdcb2), namely

$$u_o = \left[\frac{2(p_1 - p_2)}{\rho\left\{\frac{A_1^2}{C_c^2 A_o^2} - 1\right\}}\right]^{\frac{1}{2}} \quad \text{(Kdcb4)}$$

In practice orifice flow meters should be calibrated in order to provide accurate measurement.

Pitot Tubes

A Pitot tube is a small probe that is inserted into a flow in order to measure the velocity of the flow at that point. It is based on the principle that on the surface of an object in a flow there is always a point on the surface of the object facing the oncoming flow where the flow is brought to rest and the velocity at that point is zero. That point is called a **stagnation point**. It follows from Bernoulli's equation that the pressure at that stagnation point, p_s, is given by

$$p_s = p_\infty + \frac{1}{2}\rho u_\infty^2 \quad \text{(Kdcb5)}$$

where p_∞ and u_∞ are the pressure and velocity in the flow at the same elevation but some distance upstream. If a pressure tap is located at the stagnation point in order to measure p_s and if, in addition, the static pressure, p_∞ were measured then the velocity, u_∞, can be obtained from

$$u_\infty = \left\{\frac{2(p_s - p_\infty)}{\rho}\right\}^{\frac{1}{2}} \quad \text{(Kdcb5)}$$

If the probe is very small compared with the distances over which the pressure and velocity change in the flow then p_∞ and u_∞ can be assumed to be the pressure and velocity at the point where the probe is inserted.

It transpires that the pressure tap hole at the stagnation point can be large compared with the size of

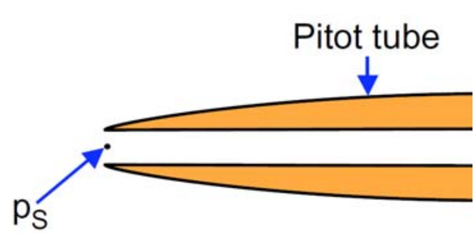

Figure 4: Pitot Tube.

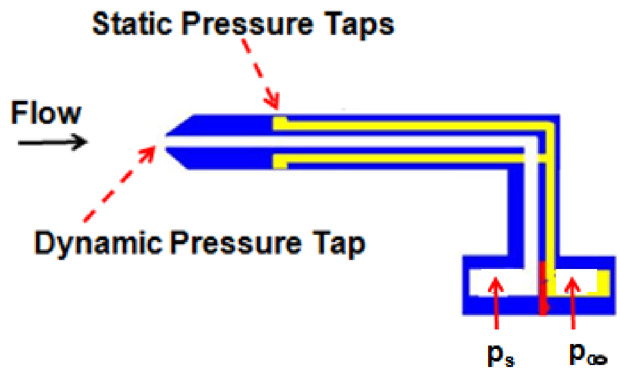

Figure 5: Pitot Tube.

Figure 6: A Pitot tube mounted on an aircraft wing.

the object or probe and (see, for example, Figure 4) yet still record the stagnation point pressure when connected to a manometer or transducer. However this is usually combined with a static pressure tap along the side of the probe (as depicted in Figure 5) so that the difference $(p_s - p_\infty)$ can be directly measured by a manometer or a differential pressure transducer. Figure 6 shows a Pitot tube mounted on an aircraft wing in order to measure the wind speed.

Variations on the Pitot Tube

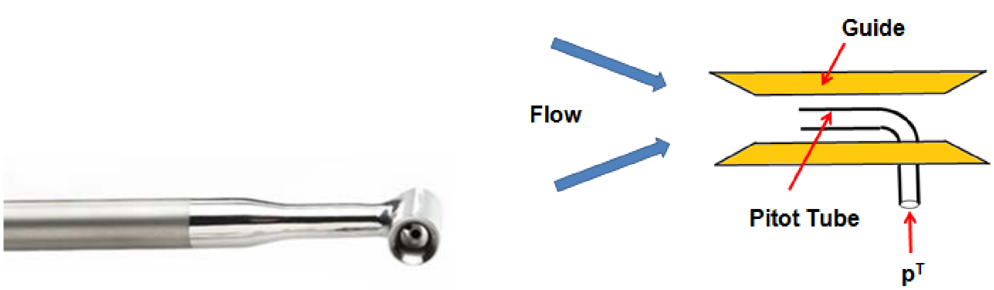

Figure 7: Kiel probe example and concept.

There are many variations on the Pitot tube that are widely used for a variety of purposes. Sometimes one cannot be very sure of the direction of the flow at the desired measurement location and a modification is need to ensure that the probe directly faces the oncoming flow since it will not function accurately otherwise. As shown in Figure 7 a so-called **Kiel probe** has a guide surrounding the Pitot tube to ensure that the flow impacts directly even though the upstream flow direction may be uncertain or variable.

Other geometric variations include flattened pitot tubes designed to get as close to a wall as possible, usually in order to measure the velocity gradient normal to the wall and therefore, knowing the viscosity, the shear stress at the wall (since by the no slip condition the velocity at the wall is zero). We conclude this section with just one other example, namely the yaw probe.

Yaw Probe

The yaw probe is another variation on the Pitot tube designed to measure the direction of the flow. Though there are a number of geometric variants, the principle is best demonstrated using the spherical yaw probe depicted in Figure 8 where the static pressures measured at two locations, A and B, on the surface of the sphere can be used to measure the inclination, α of the oncoming flow. Assuming that the pressure and

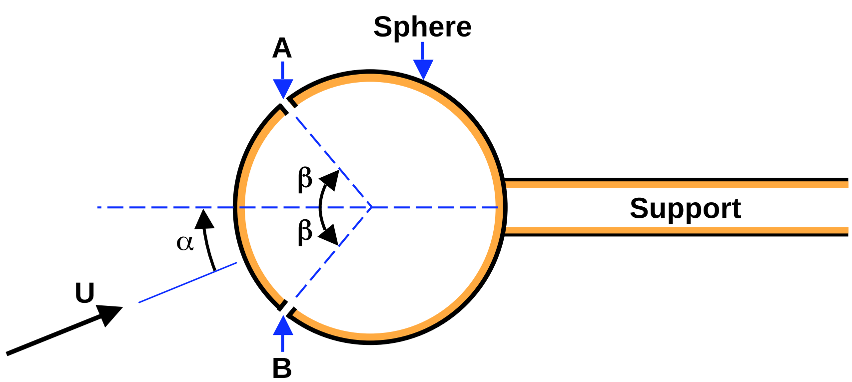

Figure 8: Diagram of a spherical yaw probe.

velocity distributions near the front stagnation point on the sphere can be approximated by the potential flow solution, it follows that the velocity on the surface of the sphere varies with angular position, θ, like $(3U \sin\theta)/2$ where U is the velocity of the oncoming flow and $\theta$is the angle measured from the front stagnation point. The point A has $\theta = \beta + \alpha$ while the point B has $\theta = \beta - \alpha$. Denoting the magnitude of the velocities at these two points by u_A and u_B it follows that

$$u_A \;=\; \frac{3U}{2}\sin(\beta+\alpha) \quad \text{and} \quad u_B \;=\; \frac{3U}{2}\sin(\beta-\alpha) \tag{Kdcb6}$$

Then, using Bernoulli's equation to relate the pressure, p_A, at A to the pressure, p_B, at B (neglecting any effect of gravity):

$$p_A + \frac{1}{2}\rho u_A^2 = p_B + \frac{1}{2}\rho u_B^2 \tag{Kdcb7}$$

and substituting for u_A and u_B into equation (Kdcb6) yields

$$p_A - p_B = \frac{9\rho U^2}{8}\left[(\sin(\beta-\alpha))^2 - (\sin(\beta+\alpha))^2\right] \tag{Kdcb8}$$

After manipulation of the trigonometric functions this yields the following relation

$$\sin 2\alpha = -\frac{8(p_A - p_B)}{9\rho U^2 \sin 2\beta} \tag{Kdcb9}$$

which allows calculation of the flow inclination α from the measurement of $(p_A - p_B)$ provided the velocity U is known. Two other pressure taps located in similar locations to those at A and B but in a plane at right angles to the plane of Figure 8 would allow a similar measurement of the flow angle in that plane. Thus four pressure taps equally distributed one the surface of the sphere allow determination in three dimensions of the direction of oncoming flow.

A slightly more sophisicated version of this is used on modern aircraft to measure the wind speed and the orientation of the aircraft to the oncoming flow. In this application the static pressure taps are replaced by pitot tubes at equidistant locations on the rounded nose of the aircraft (these are plainly visible on passenger aircraft). These four pitot tubes allow measurement of the velocity and well as the orientations (yaw and pitch) of the aircraft.