Continuum Equations for Conservation of Momentum

Continuing with the development of the differential equations, the next step is to apply the momentum principle to the elemental volume. Prior to doing so we make some minor modifications to that control volume in order to avoid some potential difficulties. Specifically we deform the bounding surfaces so that they never cut through disperse phase particles but everywhere are within the continuous phase. Since it is already assumed that the dimensions of the particles are very small compared with the dimensions of the control volume, the required modification is correspondingly small. It is possible to proceed without this modification but several complications arise. For example, if the boundaries cut through particles, it would then be necessary to determine what fraction of the control volume surface is acted upon by tractions within each of the phases and to face the difficulty of determining the tractions within the particles. Moreover, we shall later need to evaluate the interacting force between the phases within the control volume and this is complicated by the issue of dealing with the parts of particles intersected by the boundary.

Now proceeding to the application of the momentum theorem for either the disperse ($N = D$) or continuous phase ($N = C$), the flux of momentum of the N component in the k direction through a side perpendicular to the i direction is $\rho_N j_{Ni} u_{Nk}$ and hence the net flux of momentum (in the k direction) out of the elemental volume is $\partial(\rho_N \alpha_N u_{Ni} u_{Nk})/\partial x_i$. The rate of increase of momentum of component N in the k direction within the elemental volume is $\partial(\rho_N \alpha_N u_{Nk})/\partial t$. Thus using the momentum conservation principle, the net force in the k direction acting on the component N in the control volume (of unit volume), $\mathcal{F}^T_{Nk}$, must be given by

$$\mathcal{F}^T_{Nk} = \frac{\partial}{\partial t}\left(\rho_N \alpha_N u_{Nk}\right) + \frac{\partial}{\partial x_i}\left(\rho_N \alpha_N u_{Ni} u_{Nk}\right) \qquad \text{(Nbe1)}$$

It is more difficult to construct the forces, $\mathcal{F}^T_{Nk}$ in order to complete the equations of motion. We must include body forces acting within the control volume, the force due to the pressure and viscous stresses on the exterior of the control volume, and, most particularly, the force that each component imposes on the other components within the control volume.

The first contribution is that due to an external force field on the component N within the control volume. In the case of gravitational forces, this is clearly given by

$$\alpha_N \rho_N g_k \qquad \text{(Nbe2)}$$

where g_k is the component of the gravitational acceleration in the k direction (the direction of g is considered vertically downward).

The second contribution, namely that due to the tractions on the control volume, differs for the two phases because of the small deformation discussed above. It is zero for the disperse phase. For the continuous phase we define the stress tensor, σ_{Cki}, so that the contribution from the surface tractions to the force on that phase is

$$\frac{\partial \sigma_{Cki}}{\partial x_i} \qquad \text{(Nbe3)}$$

For future purposes it is also convenient to decompose σ_{Cki} into a pressure, $p_C = p$, and a deviatoric stress, σ^D_{Cki}:

$$\sigma_{Cki} = -p\delta_{ki} + \sigma^D_{Cki} \qquad \text{(Nbe4)}$$

where δ_{ki} is the Kronecker delta such that $\delta_{ki} = 1$ for $k = i$ and $\delta_{ij} = 0$ for $k \neq i$.

The third contribution to $\mathcal{F}^T_{Nk}$ is the force (per unit total volume) imposed *on* the component N *by* the other components within the control volume. We write this as $\mathcal{F}_{Nk}$ so that the Individual Phase Momentum Equation (IPME) becomes

$$\frac{\partial}{\partial t}(\rho_N \alpha_N u_{Nk}) + \frac{\partial}{\partial x_i}(\rho_N \alpha_N u_{Ni} u_{Nk})$$
$$= \alpha_N \rho_N g_k + \mathcal{F}_{Nk} - \delta_N \left\{ \frac{\partial p}{\partial x_k} - \frac{\partial \sigma^D_{Cki}}{\partial x_i} \right\} \qquad \text{(Nbe5)}$$

where $\delta_D = 0$ for the disperse phase and $\delta_C = 1$ for the continuous phase.

Thus we identify the second of the interaction terms, namely the *force interaction*, $\mathcal{F}_{Nk}$. Note that, as in the case of the mass interaction $\mathcal{I}_N$, it must follow that

$$\sum_N \mathcal{F}_{Nk} = 0 \qquad \text{(Nbe6)}$$

In disperse flows it is often useful to separate $\mathcal{F}_{Nk}$ into two components, one due to the pressure gradient in the continuous phase, $-\alpha_D \partial p/\partial x_k$, and the remainder, $\mathcal{F}'_{Dk}$, due to other effects such as the relative motion between the phases. Then

$$\mathcal{F}_{Dk} = -\mathcal{F}_{Ck} = -\alpha_D \frac{\partial p}{\partial x_k} + \mathcal{F}'_{Dk} \qquad \text{(Nbe7)}$$

The IPME (Nbe5) are frequently used in a form in which the terms on the left hand side are expanded and use is made of the continuity equation (Nbb2). In single phase flow this yields a Lagrangian time derivative of the velocity on the left hand side. In the present case the use of the continuity equation results in the appearance of the mass interaction, $\mathcal{I}_N$. Specifically, one obtains

$$\rho_N \alpha_N \left\{ \frac{\partial u_{Nk}}{\partial t} + u_{Ni} \frac{\partial u_{Nk}}{\partial x_i} \right\}$$
$$= \alpha_N \rho_N g_k + \mathcal{F}_{Nk} - \mathcal{I}_N u_{Nk} - \delta_N \left\{ \frac{\partial p}{\partial x_k} - \frac{\partial \sigma^D_{Cki}}{\partial x_i} \right\} \qquad \text{(Nbe8)}$$

Viewed from a Lagrangian perspective, the left hand side is the normal rate of increase of the momentum of the component N; the term $\mathcal{I}_N u_{Nk}$ is the rate of increase of the momentum in the component N due to the gain of mass by that phase.

If the momentum equations (Nbe5) for each of the components are added together the resulting Combined Phase Momentum Equation (CPME) becomes

$$\frac{\partial}{\partial t}\left(\sum_N \rho_N \alpha_N u_{Nk} \right) + \frac{\partial}{\partial x_i}\left(\sum_N \rho_N \alpha_N u_{Ni} u_{Nk} \right)$$
$$= \rho g_k - \frac{\partial p}{\partial x_k} + \frac{\partial \sigma^D_{Cki}}{\partial x_i} \qquad \text{(Nbe9)}$$

Note that this equation (Nbe9) will only reduce to the equation of motion for a single phase flow in the absence of relative motion, $u_{Ck} = u_{Dk}$. Note also that, in the absence of any motion (when the deviatoric stress is zero), equation (Nbe9) yields the appropriate hydrostatic pressure gradient $\partial p/\partial x_k = \rho g_k$ based on the mixture density, ρ.

Another useful limit is the case of uniform and constant sedimentation of the disperse component (volume fraction, $\alpha_D = \alpha = 1 - \alpha_C$) through the continuous phase under the influence of gravity. Then equation (Nbe5) yields

$$0 = \alpha \rho_D g_k + \mathcal{F}_{Dk}$$

$$0 = \frac{\partial \sigma_{Cki}}{\partial x_i} + (1-\alpha)\rho_C g_k + \mathcal{F}_{Ck} \qquad \text{(Nbe10)}$$

But $\mathcal{F}_{Dk} = -\mathcal{F}_{Ck}$ and, in this case, the deviatoric part of the continuous phase stress should be zero (since the flow is a simple uniform stream) so that $\sigma_{Ckj} = -p$. It follows from equation (Nbe10) that

$$\mathcal{F}_{Dk} = -\mathcal{F}_{Ck} = -\alpha \rho_D g_k \quad \text{and} \quad \partial p / \partial x_k = \rho g_k \qquad \text{(Nbe11)}$$

or, in words, the pressure gradient is hydrostatic.

Finally, note that the equivalent one-dimensional or duct flow form of the IPME is

$$\frac{\partial}{\partial t}\left(\rho_N \alpha_N u_N\right) + \frac{1}{A}\frac{\partial}{\partial x}\left(A \rho_N \alpha_N u_N^2\right) = -\delta_N \left\{ \frac{\partial p}{\partial x} + \frac{P \tau_w}{A} \right\} + \alpha_N \rho_N g_x + \mathcal{F}_{Nx} \qquad \text{(Nbe12)}$$

where, in the usual pipe flow notation, $P(x)$ is the perimeter of the cross-section and τ_w is the wall shear stress. In this equation, $A\mathcal{F}_{Nx}$ is the force imposed on the component N in the x direction by the other components per unit length of the duct. A sum over the constituents yields the combined phase momentum equation for duct flow, namely

$$\frac{\partial}{\partial t}\left(\sum_N \rho_N \alpha_N u_N\right) + \frac{1}{A}\frac{\partial}{\partial x}\left(A \sum_N \rho_N \alpha_N u_N^2\right) = -\frac{\partial p}{\partial x} - \frac{P \tau_w}{A} + \rho g_x \qquad \text{(Nbe13)}$$

and, when all phases travel at the same velocity, $u = u_N$, this reduces to

$$\frac{\partial}{\partial t}\left(\rho u\right) + \frac{1}{A}\frac{\partial}{\partial x}\left(A \rho u^2\right) = -\frac{\partial p}{\partial x} - \frac{P \tau_w}{A} + \rho g_x \qquad \text{(Nbe14)}$$