Finite Difference Methods

The basis of the finite difference method is the use of Taylor's series to find algebraic approximations for each of the terms in the partial differential equations and the boundary conditions governing the fluid flow under investigation. These approximations will involve the values of the flow and state variables at each of the points in the grid and the values of the flow and state variables at the surrounding points. Clearly the details of the appropriate finite difference method will depend heavily on the type of flow to be solved and the associated mathematical character of the equations governing that flow. For example, there are major differences in the methods used for subsonic flow and elliptic partial differential equations as opposed to supersonic flows and hyperbolic equations. When the mathematical character is different in different regions of the flow as it is for transonic flow, the challenges become even more complicated. We begin here with a brief summary of finite difference methods for elliptic partial differential equations.

Consider, for example, that we wish to solve the equation for planar potential flow, namely

$$\nabla^2\phi \;=\; 0 \tag{Oc1}$$

on a rectangular, Cartesian grid, a localized fraction of which is depicted in Figure 1. The point labeled

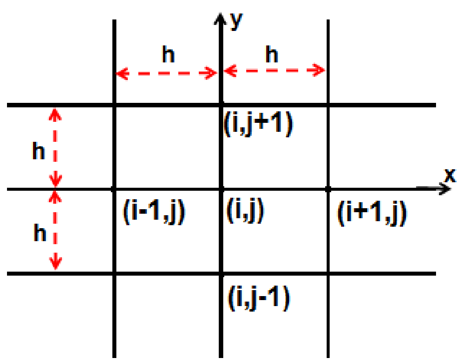

Figure 1: A simple grid element for planar flow.

(i,j) is a general point within the flow field and sufficiently far from the boundaries so that the surrounding points, $(i+1,j)$, $(i,j+1)$, $(i-1,j)$ and $(i,j-1)$ are also within the flow field. A rectangular grid with a **grid spacing** of h has been chosen (much more on this choice later) and it is assumed that h is small compared with the distance over which there are significant changes in the flow variable, ϕ. Then we use Taylor's series to write the following relations between the value of ϕ at the point (i,j) (denoted by $\phi_{i,j}$) and the values at the surrounding points:

$$\phi_{i+1,j} \;=\; \phi_{i,j} + h\left\{\frac{\partial\phi}{\partial x}\right\}_{i,j} + \frac{h^2}{2}\left\{\frac{\partial^2\phi}{\partial x^2}\right\}_{i,j} + \frac{h^3}{6}\left\{\frac{\partial^3\phi}{\partial x^3}\right\}_{i,j} + \frac{h^4}{24}\left\{\frac{\partial^4\phi}{\partial x^4}\right\}_{i,j} + O(h^5) \tag{Oc2}$$

$$\phi_{i-1,j} \;=\; \phi_{i,j} - h\left\{\frac{\partial\phi}{\partial x}\right\}_{i,j} + \frac{h^2}{2}\left\{\frac{\partial^2\phi}{\partial x^2}\right\}_{i,j} - \frac{h^3}{6}\left\{\frac{\partial^3\phi}{\partial x^3}\right\}_{i,j} + \frac{h^4}{24}\left\{\frac{\partial^4\phi}{\partial x^4}\right\}_{i,j} - O(h^5) \tag{Oc3}$$

$$\phi_{i,j+1} \;=\; \phi_{i,j} + h\left\{\frac{\partial\phi}{\partial y}\right\}_{i,j} + \frac{h^2}{2}\left\{\frac{\partial^2\phi}{\partial y^2}\right\}_{i,j} + \frac{h^3}{6}\left\{\frac{\partial^3\phi}{\partial y^3}\right\}_{i,j} + \frac{h^4}{24}\left\{\frac{\partial^4\phi}{\partial y^4}\right\}_{i,j} + O(h^5) \tag{Oc4}$$

$$\phi_{i,j-1} = \phi_{i,j} - h\left\{\frac{\partial\phi}{\partial y}\right\}_{i,j} + \frac{h^2}{2}\left\{\frac{\partial^2\phi}{\partial y^2}\right\}_{i,j} - \frac{h^3}{6}\left\{\frac{\partial^3\phi}{\partial y^3}\right\}_{i,j} + \frac{h^4}{24}\left\{\frac{\partial^4\phi}{\partial y^4}\right\}_{i,j} - O(h^5) \quad \text{(Oc5)}$$

where the subscript i, j refers to evaluation at the point (i, j). Adding the relations (Oc2) and (Oc3) and then the relations (Oc4) and (Oc5) yields the following finite difference expressions for the second derivatives:

$$\left\{\frac{\partial^2\phi}{\partial x^2}\right\}_{i,j} = \frac{\phi_{i+1,j} + \phi_{i-1,j} - 2\phi_{i,j}}{h^2} + \frac{h^2}{12}\left\{\frac{\partial^4\phi}{\partial x^4}\right\}_{i,j} + O(h^3) \quad \text{(Oc6)}$$

$$\left\{\frac{\partial^2\phi}{\partial y^2}\right\}_{i,j} = \frac{\phi_{i,j+1} + \phi_{i,j-1} - 2\phi_{i,j}}{h^2} + \frac{h^2}{12}\left\{\frac{\partial^4\phi}{\partial y^4}\right\}_{i,j} + O(h^3) \quad \text{(Oc7)}$$

and so Laplace's equation for the velocity potential can be written in the finite difference form

$$\phi_{i+1,j} + \phi_{i,j+1} + \phi_{i-1,j} + \phi_{i,j-1} - 4\phi_{i,j} = \frac{h^4}{12}\left[\left\{\frac{\partial^4\phi}{\partial x^4}\right\}_{i,j} + \left\{\frac{\partial^4\phi}{\partial y^4}\right\}_{i,j}\right] + O(h^5) \quad \text{(Oc8)}$$

Therefore a finite difference approximation to Laplace's equation is

$$\phi_{i+1,j} + \phi_{i,j+1} + \phi_{i-1,j} + \phi_{i,j-1} - 4\phi_{i,j} = 0 \quad \text{(Oc9)}$$

with an error at each point of order

$$\frac{h^4}{12}\left[\left\{\frac{\partial^4\phi}{\partial x^4}\right\}_{i,j} + \left\{\frac{\partial^4\phi}{\partial y^4}\right\}_{i,j}\right] \quad \text{(Oc10)}$$

Note that this implies that the value of $\phi_{i,j}$ should be the arithmetic average of the values of ϕ at the four surrounding points. Note also that h must be small enough that the error is acceptable. Of course the error can be further reduced by involving grid points further away, for example the points $(i+2, j)$ and $(i-2, j)$ in the case of the higher order x derivatives. However, this adds complexity and it is often better to improve accuracy by decreasing the grid spacing h. The above finite difference approximations will also be different if the grid spacing is different in the various directions; then modified expressions follow from the modified Taylor series expansions.

The above relations apply to grid points within the flow field. A modified approach is needed for points on the boundary of the flow and these modified finite difference approximations necessarily involve the appropriate boundary condition. As a simple example consider the grid point (i, j) on the boundary depicted in Figure 2. Equations (Oc2) to (Oc5) still apply but now we recognize that in equations (Oc2)

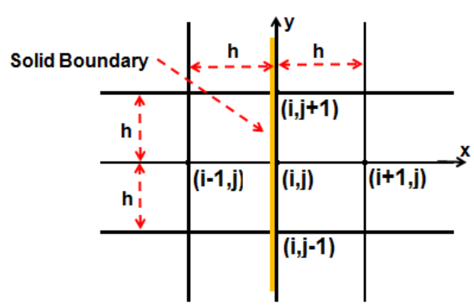

Figure 2: A grid point (i, j) on a solid boundary.

and (Oc3) the derivative $(\partial\phi/\partial x)_{i,j}$ must be zero since this is equal to the velocity normal to the wall at that point and that velocity must be zero. This boundary condition therefore allows us to write equations (Oc2) and (Oc3) as

$$\phi_{i+1,j} - \phi_{i,j} = \frac{h^2}{2}\left\{\frac{\partial^2\phi}{\partial x^2}\right\}_{i,j} + \frac{h^3}{6}\left\{\frac{\partial^3\phi}{\partial x^3}\right\}_{i,j} + \frac{h^4}{24}\left\{\frac{\partial^4\phi}{\partial x^4}\right\}_{i,j} + O(h^5) \tag{Oc11}$$

$$\phi_{i-1,j} - \phi_{i,j} = \frac{h^2}{2}\left\{\frac{\partial^2\phi}{\partial x^2}\right\}_{i,j} - \frac{h^3}{6}\left\{\frac{\partial^3\phi}{\partial x^3}\right\}_{i,j} + \frac{h^4}{24}\left\{\frac{\partial^4\phi}{\partial x^4}\right\}_{i,j} - O(h^5) \tag{Oc12}$$

and subtracting these expressions

$$\phi_{i+1,j} - \phi_{i-1,j} = \frac{h^3}{3}\left\{\frac{\partial^3\phi}{\partial x^3}\right\}_{i,j} + O(h^5) \tag{Oc13}$$

Therefore with an error of order h^4 we have that

$$\phi_{i+1,j} = \phi_{i-1,j} + O(h^4) \tag{Oc13}$$

Consequently we can either use an image point $(i-1,j)$ whose value is always set equal to the value at $(i+1,j)$ or, equivalently, we can simply use the value at $(i+1,j)$ in equation (Oc9) or whenever else a value for $(i-1,j)$ is needed. Note that the error is of slightly higher order when we do so but this can be compensated for by adjusting the grid spacing, h.

Clearly, then, in a corner such as that shown in Figure 3 we would not only use the value at $(i+1,j)$ whenever a value for $(i-1,j)$ is needed but also use the value at $(i,j+1)$ whenever a value for $(i,j-1)$ is needed. A treatment like this will work for an acute-angle corner. However, more serious problems

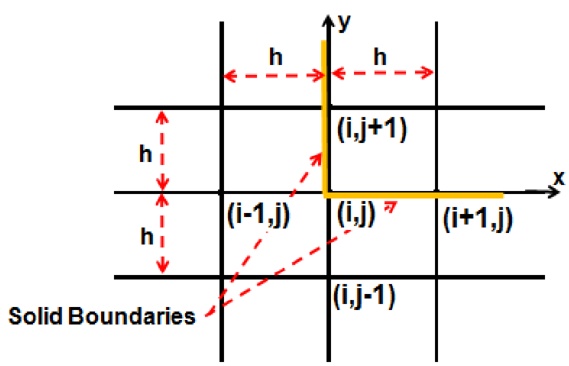

Figure 3: A grid point in a rectangular corner.

can arise with obtuse-angled corners that project into the flow, particularly in potential flows where the velocity at such a corner may tend to infinity as the corner is approached. Singular points like this require special treatment whenever they occur since the derivatives in the Taylor's series expansions may become very large and therefore the errors in the finite difference approximations will no longer be acceptable. In such cases, the first step must be to identify the analytical form of the singular behavior at the projecting corner; in planar potential flow this is relatively easy since the potential flow solution must be of the form, $Cr^n\{\sin(n(\theta-\theta_0))$ where (r,θ) are polar coordinates based at the singular point, n and θ are known from the inclination of the two converging walls and only the constant C is unknown. Thus, apart from the constant C, the analytical form of the velocity potential near the projecting vertex is known; we denote this by ϕ'. Then the numerical treatment is to subtract this singular behavior from the total potential, ϕ, and represent the non-singular remainder by $\Delta\phi$ so that

$$\phi = \phi' + \Delta\phi \tag{Oc14}$$

so that the remainder, $\Delta\phi$, is well behaved and can be treated by finite difference approximations similar to those in the rest of the flow field. Both ϕ' and $\Delta\phi$ must satisfy the governing partial differential equation and the constant C must be adjusted to match the finite difference solution at the neighboring grid points. An illustration of how this is accomplished is given in the following sample numerical solution.

Another common boundary is one in which the flow enters the gridded region. At such a boundary a possible condition would be to set ϕ to a constant value along that boundary. This implies that the velocity tangential to the boundary is zero and therefore that the flow is normal to the boundary. If that were the inflow boundary then one might have another boundary where the flow exits the gridded region. Then one might set the value of ϕ on that boundary to a different, uniform value. Note that since the equations and boundary conditions for potential flow are linear in ϕ, the final solution can be uniformly magnified or condensed in order to yield the desired inflow or discharge from the gridded region. The attached example will demonstrate how this can be carried out.

The final assemblage of finite difference equations must be equal in number to the number of unknown nodal values, N. There are several alternative ways in which to solve this set of equations. If the equations are all linear as they are in the attached example then one possible approach is to assemble those equations in matrix form. In the case described above, this might take the form

$$[A]\{\phi\} = \{C\} \tag{Oc15}$$

where $[A]$ is a $N \times N$ matrix of known coefficients and $\{C\}$ is a vector of known constants. In theory the matrix $[A]$ could be inverted and the solution obtained as

$$\{\phi\} = [A]^{-1}\{C\} \tag{Oc16}$$

The problem is that N may be such a large number that the inversion may be excessively time-consuming. Moreover, since the matrix $[A]$ is very sparse it contains many zeroes and there are many superfluous arithmetic steps involved in the inversion process. Though there exist ways of making the inversion of such sparse matrices more efficient, they involve some complicated computer programming.

An alternative way to find a solution of the assemblage of finite difference equations is to approach that solution iteratively. This involves visiting each of the unknowns or grid points in turn and adjusting the values in the hope that the values converge to a state in which all the finite difference equations are satisfied. This process begins by assigning a guessed value to each of the unknowns. One would then visit each grid point value and adjust or relax its value so as to satisfy the pertinent equation at that grid point. This process is often termed a **relaxation method** since one momentarily satifies the equation by adjusting the value of the unknown at that point. Of course when you then move on to a neighboring grid point and "relax" that point, the equation at the preceding neighboring point may no longer be satisfied. However, the hope is that if you cycle through all the grid points enough times, all the grid point values settle down into a state in which all the equations are satisfied. The advantage of these methods is that the programming can be very simple and quite efficient. There are two classic variations. In the first, known as the Jacobi method, a complete set of the unknowns are stored and one determines the adjustments based entirely on that "old" set of values. In the second, known as the Gauss-Seidel method, the adjustments are based on the most recently obtained values; this avoids the need to store one complete set of values.

In the preceding example, equation (Oc9) suggests that the relaxation at a general grid point should consist of adjusting the value of $\phi_{i,j}$ to

$$\phi'_{i,j} = \frac{1}{4}\{\phi_{i+1,j} + \phi_{i,j+1} + \phi_{i-1,j} + \phi_{i,j-1}\} \tag{Oc17}$$

where we have used the prime to distinguish the above value from the preceding value (unprimed). While such a method will usually work, it is preferable to use an over- or under-relaxation factor which we will denote by α and to relax the grid point (i, j) by setting the new value to be

$$(1-\alpha)\phi_{i,j} + \alpha\phi'_{i,j} \tag{Oc18}$$

Then a value of α greater than one will impose an adjustment greater than that suggested by equation (Oc16) in anticipation that the adjustment during the next cycle will be in the same direction and therefore convergence toward a steady state will be improved. On the other hand a value of α less than one might prevent divergence from occuring. In the example of the following section values of α of the order of 1.5 improve convergence.