Problem 116B

In spherical coordinates, (r,θ,ϕ), the equations of motion for an inviscid fluid, Euler's equations, become:

$$\rho\left\{\frac{Du_r}{Dt}-\frac{u_\theta^2+u_\phi^2}{r}\right\}=-\frac{\partial p}{\partial r}+f_r$$

$$\rho\left\{\frac{Du_\theta}{Dt}+\frac{u_\theta u_r}{r}-\frac{u_\phi^2\cot\theta}{r}\right\}=-\frac{1}{r}\frac{\partial p}{\partial\theta}+f_\theta$$

$$\rho\left\{\frac{Du_\phi}{Dt}+\frac{u_\phi u_r}{r}+\frac{u_\theta u_\phi\cot\theta}{r}\right\}=-\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi}+f_\phi$$

where u_r, u_θ, u_ϕ are the velocities in the r,θ,ϕ directions, p is the pressure, ρ is the fluid density and f_r, f_θ, f_ϕ are the body force components. The Lagrangian or material derivative is

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+u_r\frac{\partial}{\partial r}+\frac{u_\theta}{r}\frac{\partial}{\partial\theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

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For an incompressible fluid the equation of continuity in spherical coordinates is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2u_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(u_\theta\sin\theta\right)+\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}=0$$

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An underwater explosion creates a purely radial flow ($u_\theta = u_\phi = 0$ and $\partial/\partial\theta = 0$ and $\partial/\partial\phi = 0$) in water surrounding a bubble whose radius, denoted by $R(t)$, is increasing with time. Since the u_r velocity at the surface of the bubble must be equal to dR/dt show that the equation of continuity requires that

$$u_r=\frac{R^2}{r^2}\frac{dR}{dt}$$

Assume that the water is incompressible. Also note that, since R is a function only of time, there is no ambiguity about its time derivative and hence dR/dt is just an ordinary time derivative.

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Now use the equations of motion to find the pressure, $p(r,t)$, at any position, r, in the water. Neglect all body forces. One integration step has to be performed which introduces an integration constant; this can be evaluated by assuming the pressure far from the bubble ($r\to\infty$) is known (denoted by p_∞).

Finally show that, if one neglects surface tension so that the pressure in the bubble, p_B, is the same as the pressure in the water at $r=R$, then

$$p_B-p_\infty=\rho\left\{R\frac{d^2R}{dt^2}+\frac{3}{2}\left(\frac{dR}{dt}\right)^2\right\}$$

This is known as the Rayleigh equation for bubble dynamics.