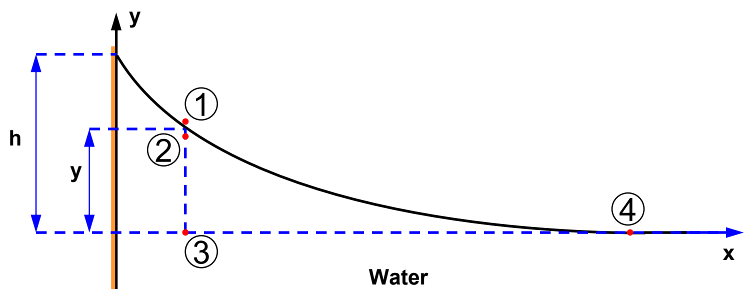
Solution to Problem 108D

Consider the pressure at four points:

1. Any point in the air just above the surface
2. The point in the fluid just below point 1
3. The point vertically below point 2 on the horizontal line
4. A point far off to the right where the surface is horizontal. This point is on the same horizontal line as point 3.



The pressure at each point is:

$$\begin{aligned} p_1 &= p_{atm} = p_2 + S\frac{d^2y}{dx^2} \\ p_2 &= p_3 - \rho g y \\ p_3 &= p_{atm} \\ p_4 &= p_{atm} \end{aligned}$$

where S is the surface tension and $y = y(x)$ is the equation of the meniscus. At point 4 as $r \to \infty$, $S\frac{d^2y}{dx^2} = \frac{S}{r} \to 0$. From these equations,

$$\begin{aligned} p_1 = p_{atm} &= p_2 + S\frac{d^2y}{dx^2} \\ &= (p_3 - \rho g y) + S\frac{d^2y}{dx^2} \\ &= p_{atm} - \rho g y + S\frac{d^2y}{dx^2} \\ \to \frac{d^2y}{dx^2} &= \frac{\rho g y}{S} \end{aligned}$$

The solution for the second order differential equation has the form:

$$y(x) = Ae^{\alpha x} + Be^{-\alpha x} \tag{1}$$

where $\alpha = \sqrt{\frac{\rho g}{S}}$ and the constants A and B can be found using the boundary conditions

$$\begin{aligned} @\ x = 0: &\qquad \frac{dy}{dx} = \tan(\theta + \pi/2) = \cot(\theta) \\ @\ x = \infty: &\qquad y = 0 \\ \to A = 0, &\qquad B = \sqrt{\frac{S}{\rho g}}\cot\theta \end{aligned}$$

$$\therefore \qquad y = \sqrt{\frac{S}{\rho g}} \cot\theta \; e^{-\left(\sqrt{\frac{\rho g}{S}}\right)x}$$

and

$$h = \sqrt{\frac{S}{\rho g}} \cot\theta$$