Solution to Problem 117A

Euler's momentum equations for the inviscid planar flow of an incompressible fluid under the action of conservative body forces ($f_x = \partial F/\partial x$ and $f_y = \partial F/\partial y$ where F is the body force potential) are:

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = -\frac{\partial p}{\partial x} + \frac{\partial F}{\partial x}$$
$$\rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right] = -\frac{\partial p}{\partial y} + \frac{\partial F}{\partial y}$$

and, since the flow is incompressible, the continuity equation is:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

To eliminate the pressure from the two momentum equations, take $\partial/\partial y$ of the first or x momentum equation and $\partial/\partial x$ of the second:

$$\rho\left[\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}\right) + \frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(v\frac{\partial u}{\partial y}\right)\right] = -\frac{\partial^2 p}{\partial x\partial y} + \frac{\partial^2 F}{\partial x\partial y}$$
$$\rho\left[\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}\right) + \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial x}\left(v\frac{\partial v}{\partial y}\right)\right] = -\frac{\partial^2 p}{\partial y\partial x} + \frac{\partial^2 F}{\partial y\partial x}$$

Subtract the two equations and group terms:

$$\rho\left[\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) - \frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\partial v}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right] = 0$$

Finally, substitute in $\omega = \partial u/\partial y - \partial v/\partial x$ and using the continuity equation, $\partial u/\partial x + \partial v/\partial y = 0$ to obtain:

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = 0$$

This equation tells us that $D\omega/Dt = 0$ and therefore the vorticity associated with a particular fluid element does not change as the fluid element moves along in the flow.