Solution to Problem 119A

Note that for incompressible planar potential flow, the velocity potential ϕ is given by:

$$\vec{u} = \nabla \phi$$

and the velocity components are given by the Cauchy-Riemann equations:

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

Part (a)

The streamfunction is given as:

$$\psi = Axy$$

with the related velocity components:

$$u = Ax \quad \text{and} \quad v = -Ay$$

First we have to check that the flow is indeed irrotational:

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

The velocity potential ϕ is calculated using the Cauchy-Riemann equations:

$$\phi = \int \frac{\partial \phi}{\partial y} dx = \frac{1}{2} A x^2 + C(y) + E$$

and

$$\phi = \int -\frac{\partial \phi}{\partial x} dy = -\frac{1}{2} A y^2 + D(x) + E$$

In order to satisfy both equations, we set:

$$\phi(x, y) = \frac{1}{2} A \left(x^2 - y^2 \right) + E$$

The boundary condition $\phi(0, 0) = 0$ sets $E = 0$.

Part (b)

The streamfunction is given as:

$$\psi = A \left(x^2 - y^2 \right)$$

with the related velocity components:

$$u = -2Ay$$

and

$$v = -2Ax$$

First we have to check that the flow is indeed irrotational:

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -2A + 2A = 0$$

The velocity potential ϕ is calculated using the Cauchy-Riemann equations:

$$\phi = \int \frac{\partial \phi}{\partial y} dx = -2Axy + C(y) + E$$

and

$$\phi = \int -\frac{\partial \phi}{\partial x} dy = -2Axy + D(x) + E$$

In order to satisfy both equations, we set:

$$\phi(x, y) = -2Axy + E$$

The boundary condition $\phi(0, 0) = 0$ sets $E = 0$.

Part (c)

The streamfunction is given as:

$$\psi = A\left(x^2 y - \frac{1}{3} y^3\right)$$

with the related velocity components:

$$u = A\left(x^2 - y^2\right)$$

and

$$v = -2Axy$$

First we have to check that the flow is indeed irrotational:

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -2Ay + 2Ay = 0$$

The velocity potential ϕ is calculated using the Cauchy-Riemann equations:

$$\phi = \int \frac{\partial \phi}{\partial y} dx = A\left(\frac{1}{3}x^3 - y^2 x\right) + C(y) + E$$

and

$$\phi = \int -\frac{\partial \phi}{\partial x} dy = -Axy^2 + D(x) + E$$

In order to satisfy both equations, we set:

$$\phi(x, y) = A\left(\frac{1}{3}x^3 - y^2 x\right) + E$$

The boundary condition $\phi(0, 0) = 0$ sets $E = 0$.