Solution to Problem 119B:

This problem is most readily done in polar coordinates rather than Cartesian coordinates. . But here we demonstrate the solution using Cartesian coordinates and velocities. Since $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$ then

For Flow (i):

$$u = 2Ay \quad ; \quad v = -2Ax \tag{1}$$

and the magnitude of the velocity

$$(u^2+v^2)^{1/2} = 2A(x^2+y^2)^{1/2} = 2AR \tag{2}$$

For Flow (ii):

$$u = Ay/(x^2+y^2) \quad ; \quad v = -Ax/(x^2+y^2) \tag{3}$$

and the magnitude of the velocity

$$(u^2+v^2)^{1/2} = A\frac{(x^2+y^2)^{1/2}}{(x^2+y^2)} = A/R \tag{4}$$

The direction of the velocity is tangent to a circle of radius, R, about the origin since the streamlines (lines of constant ψ) are circles in both cases; the velocity is positive in the anticlockwise direction.

Notice that the magnitude of the velocity in Flow (i) is proportional to the radius, R, of the streamline and therefore the flow at large radii becomes unphysical. On the other hand the velocity of Flow (ii) is inversely proportional to R and though the velocity decays to zero at large R the flow tends to infinity at small R, near the origin and there becomes unphysical.

These two simple vortex motions are called a *FORCED VORTEX* and a *FREE VORTEX* in fluid mechanics. The former is simply the result of solid body like rotation of the fluid; there is no shear. The latter involves shear. I practice, for reasons discussed in class, an actual vortex has an *inner core* like a forced vortex in which the velocity is approximately proportional to R and an outer part that behaves like a free vortex with a velocity inversely proportional to R. In other words:

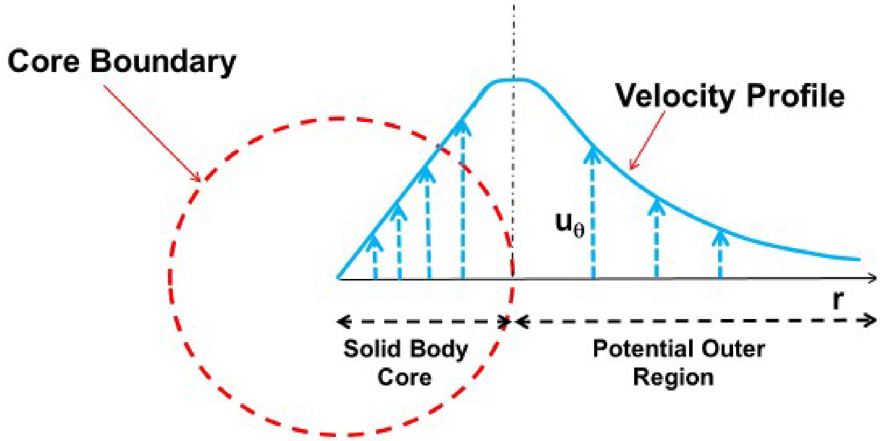
Vorticity: Since the vorticity, ω, is given by

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{5}$$

For Flow (i), $\omega = -2A - 2A = -4A$ and the forced vortex has a uniform vorticity. For Flow (ii), $\omega = 0$ and so the free vortex is irrotational.

In class we showed that the steady, irrotational flow of an incompressible and inviscid fluid is governed by Bernoulli's equation and if we neglect the effect of gravity in this planar flow, the pressure, p, is given by

$$p = \text{constant} - \frac{1}{2}(u^2+v^2) \tag{6}$$



Since Flow (ii) is irrotational, the pressure in that flow is given by

$$p = \text{constant} - \frac{\rho}{2}\frac{A^2}{R^2} \tag{7}$$

and, since $p = p_\infty$ when $R \to \infty$

$$p = p_\infty - \frac{\rho}{2}\frac{A^2}{R^2} \tag{8}$$

and the isobars are concentric circles.