Solution to Problem 120G

The flow around a cylinder of radius, R, is generated by the superposition of a uniform stream, a doublet, and a potential vortex:

$$\phi = \underbrace{Ur\cos\theta}_{\text{Uniform Stream}} \underbrace{+U\frac{R^2}{r}\cos\theta}_{\text{Doublet}} + \underbrace{\frac{\Gamma\theta}{2\pi}}_{\text{Vortex}}$$

The velocity components are therefore

$$u_r = \frac{\partial\phi}{\partial r} = U\left(1-\frac{R^2}{r^2}\right)\cos\theta$$

$$u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\left(1+\frac{R^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r}$$

and on the surface of the cylinder ($r=R$):

$$u_r|_{r=R} = U\left(1-\frac{R^2}{R^2}\right)\cos\theta = 0$$

$$u_\theta|_{r=R} = -2U\sin\theta + \frac{\Gamma}{2\pi R}$$

The radial velocity must be zero for there to be no flow through the cylinder. To find the corresponding pressure on the surface of the cylinder, we use Bernoulli's Equation (since the flow is irrotational):

$$\frac{1}{2}\rho|u(R,\theta)|^2 + p(R,\theta) = \frac{1}{2}\rho|u_\infty|^2 + p_\infty$$

$$\therefore p(R,\theta) = \frac{1}{2}\rho\left(U^2 - \left[-2U\sin\theta + \frac{\Gamma}{2\pi R}\right]^2\right) + p_\infty$$

The total force on the cylinder per unit depth normal to the sketch is given as the integral over the surface of the pressure multiplied by the area. The components of this force in the horizontal and vertical directions, the drag and lift, respectively, are then found to be:

$$D = F_x = -\int_0^{2\pi} p(R,\theta)\cos\theta R d\theta$$

$$L = F_y = -\int_0^{2\pi} p(R,\theta)\sin\theta R d\theta$$

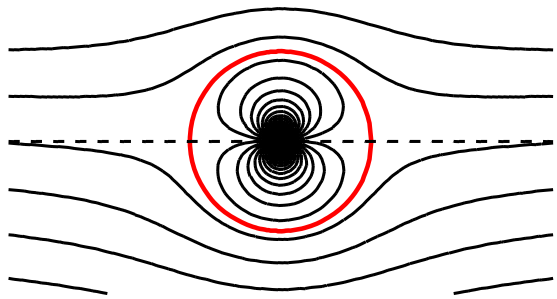
Evaluating the drag:

$$\begin{aligned} D &= -\int_0^{2\pi}\left[\frac{1}{2}\rho U^2 R\left(\cos\theta - 4\sin^2\theta\cos\theta + \frac{2\Gamma}{\pi UR}\sin\theta\cos\theta - \frac{\Gamma^2}{4\pi^2R^2U^2}\cos\theta\right) + p_\infty R\cos\theta\right]d\theta \\ &= -\left[\frac{1}{2}\rho U^2 R\left(\sin\theta\left[1-\frac{\Gamma^2}{4\pi^2R^2U^2}\right] - \frac{4}{3}\sin^3\theta + \frac{\Gamma}{\pi UR}\sin^2\theta\right) + p_\infty R\sin\theta\right]_0^{2\pi} \\ &= 0 \end{aligned}$$

since $\sin 0 = \sin 2\pi = 0$. Evaluating the lift:

$$\begin{aligned} L &= -\int_0^{2\pi}\left[\frac{1}{2}\rho U^2 R\left(\sin\theta - 4\sin^3\theta + \frac{2\Gamma}{\pi UR}\sin^2\theta - \frac{\Gamma^2}{4\pi^2R^2U^2}\sin\theta\right) + p_\infty R\sin\theta\right]d\theta \\ &= -\left[\frac{1}{2}\rho U^2 R\left(-\cos\theta\left[1-\frac{\Gamma^2}{4\pi^2R^2U^2}\right] + \frac{4}{3}\sin^2\theta\cos\theta + \frac{8}{3}cos\theta + \frac{\Gamma}{\pi UR}\theta - \frac{\Gamma}{2\pi UR}\sin 2\theta\right) - p_\infty R\cos\theta\right]_0^{2\pi} \\ &= -\rho\Gamma U \end{aligned}$$

The lack of drag could be inferred from the front-to-back symmetry of the streamlines on the cylinder and the corresponding symmetry in the magnitude of the velocity.



Since the pressure is directly coupled to the velocity through Bernoulli's equation, symmetric streamlines imply that there is no pressure imbalance front to back and thus no force. The lack of drag is known as d'Alembert's paradox. Turning to the lift, the circulation, Γ, breaks the top-to-bottom symmetry of the streamlines and creates a net force, the lift. The presence of lift based on the magnitude of the circulation is reflected in the general result known as the Kutta-Joukowski Theorem.