Solution to Problem 130C

(a) The flow must satisfy the following four boundary conditions,

1. $u = 0$ at $x = 0$,
2. $u = 0$ at $x = L$,
3. On the free surface ($y = h$),
$$v|_{y=h} = \frac{\partial h}{\partial t}$$
but for small amplitude waves, $v|_{y=h} \approx v|_{y=0}$, so the kinematic condition is
$$v|_{y=0} = \frac{\partial h}{\partial t}$$
4. The dynamic condition on the free surface, namely that the pressure is constant and is equal to the atmospheric pressure.

(b) The velocity potential is given as
$$\phi = Ae^{ky} \cos kx \sin \omega t$$
where A, k and ω are undetermined constants. The velocity in the x-direction, u, is
$$u = \frac{\partial \phi}{\partial x} = -Ake^{ky} \sin kx \sin \omega t$$
The boundary condition at $x = 0$ is automatically satisfied by the above equation, but for $u = 0$ at $x = L$,
$$kL = n\pi$$
where n is an integer. From the relationship between the wave number, k, and the wavelength, λ,
$$\lambda = \frac{2\pi}{k} = \frac{2L}{n}, \quad n = \text{integer}$$
Thus, there can be a half wave ($n = 1$), full wave ($n = 2$), etc. trapped between the walls.

(c) The velocity v in the y-direction is,
$$v = \frac{\partial \phi}{\partial y} = Ake^{ky} \cos kx \sin \omega t$$
and the kinematic condition gives
$$\frac{\partial h}{\partial t} = v|_{y=0} = Ak \cos kx \sin \omega t$$
Integrating the above equation yields
$$h(x,t) = -\frac{Ak}{\omega} \cos kx \cos \omega t$$
where the constant of integration, some unknown function $f(x)$, must be zero if the x-axis is assumed to lie at the centerline of the waves.

(d) The unsteady Bernoulli equation requires
$$\rho \frac{\partial \phi}{\partial t} + p + \frac{1}{2}\rho \left(u^2 + v^2\right) + \rho g y = \text{constant}$$

On the free surface, the dynamic condition gives that pressure p is constant. The kinetic energy terms ($\frac{1}{2}\rho u^2$ and $\frac{1}{2}\rho v^2$) are of higher order than the other terms and are thus negligible. Finally, substituting the height for y yields,

$$\rho \left.\frac{\partial \phi}{\partial t}\right|_{y=h} + \rho g h = \text{constant}$$

The small amplitude assumption allows the approximation,

$$\left.\frac{\partial \phi}{\partial t}\right|_{y=h} \simeq \left.\frac{\partial \phi}{\partial t}\right|_{y=0} = A\omega \cos kx \cos \omega t$$

which, when substituted into Bernoulli's equation yields

$$A\omega \cos kx \cos \omega t - \frac{Agk}{\omega} \cos kx \cos \omega t = \text{constant}$$

The only constant which will satisfy this equation at all times is zero and thus

$$A\omega \cos kx \cos \omega t - \frac{Agk}{\omega} \cos kx \cos \omega t = 0,$$

which yields

$$A\omega - \frac{Agk}{\omega} = 0 \quad \text{and} \quad \omega = \sqrt{gk}$$

Thus, the frequency, f ($f = \omega/2\pi$) is given by

$$f = \frac{1}{2\pi} \left(\frac{n\pi g}{L}\right)^{\frac{1}{2}}$$