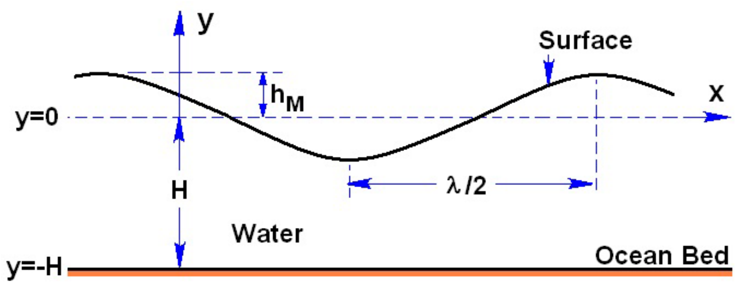
Solution to Problem 130E:

Traveling waves on an ocean of finite depth, H. The velocity potential will be of the form



$$\phi = \left\{Ae^{ny} + Be^{-ny}\right\}\sin(nx - \Omega t) \tag{1}$$

where $n = 2\pi/\lambda$ (λ is the wavelength) and Ω is the radian frequency. The speed, c, of the waves is Ω/n. The corresponding velocities are:

$$u = \frac{\partial \phi}{\partial x} = \left\{Ane^{ny} + Bne^{-ny}\right\}\cos(nx - \Omega t) \tag{2}$$

$$v = \frac{\partial \phi}{\partial y} = \left\{Ane^{ny} - Bne^{-ny}\right\}\sin(nx - \Omega t) \tag{3}$$

The boundary conditions are now applied assuming small amplitude. First the kinematic boundary condition on the free surface requires that

$$v_{y=h} \approx v_{y=0} = \left\{\frac{\partial \phi}{\partial y}\right\}_{y=0} = \frac{\partial h}{\partial t} \tag{4}$$

or

$$\frac{\partial h}{\partial t} = n(A - B)\sin(nx - \Omega t) \tag{5}$$

and therefore

$$h = \frac{n(A - B)}{\Omega}\cos(nx - \Omega t) \tag{6}$$

Secondly the boundary condition on the bottom, $y = -H$, is $v_{y=-H} = 0$ so that

$$A = Be^{2nH} \tag{7}$$

And thirdly the dynamic condition on the free surface is that the pressure, $p_{y=h}$, should be constant. But by Bernoulli's equation

$$\left(\rho\frac{\partial \phi}{\partial t} + p + \frac{1}{2}\rho\,|u|^2 + \rho g y\right)_{y=h} = \text{Constant} \tag{8}$$

Neglecting the third term in the brackets (small amplitude assumption) the dynamic condition on the free surface requires that

$$\left(\frac{\partial \phi}{\partial t}\right)_{y=0} + gh = \text{Constant} \tag{9}$$

Therefore

$$-\Omega(A+B)\cos(nx-\Omega t)+\frac{ng(A-B)}{\Omega}\cos(nx-\Omega t) \quad = \quad \text{Constant} \tag{10}$$

and this constant must be zero if this is to be true for all x and t. Therefore

$$\Omega^2 \quad = \quad gn\frac{(A-B)}{(A+B)} \quad = \quad gn\tanh(nH) \tag{11}$$

and

$$c^2 \quad = \quad \frac{\Omega^2}{n^2} \quad = \quad \frac{g}{n}\tanh(nH) \tag{12}$$

or

$$c \quad = \quad \left[\frac{g\lambda}{2\pi}\tanh\left(\frac{2\pi H}{\lambda}\right)\right]^{1/2} \tag{13}$$