Solution to Problem 130F

As in the course notes, we solve Laplace's equation ($\nabla^2\phi = 0$) with appropriate boundary condtions. The conditions in the horizontal direction state that there is no flow through the side walls:

$$BC\#1: \quad u(0,y,t) = 0$$

$$BC\#2: \quad u(L,y,t) = 0$$

In the vertical direction, there will be no flow through the bottom ($y = -H$). The condition at the free surface is more involved. Here we use the kinematic and dynamic (from the unsteady Bernoulli equation) conditions to form the fourth boundary condtion:

$$BC\#3: \quad v(x,-H,t) = 0$$

$$BC\#4a: \quad \frac{\partial h}{\partial t} = v(x,h,t) \approx v(x,0,t)$$

$$Bc\#4b: \quad \left.\frac{\partial \phi}{\partial t}\right|_{y=0} + gh = const$$

A solution for standing waves is given by:

$$\phi = \left(Ae^{ky} + Be^{-ky}\right)\cos kx \sin\omega t$$

To apply the first two boundary conditions, we calculate the velocity in the horizontal direction:

$$u = \frac{\partial \phi}{\partial x} = -k\left(Ae^{ky} + Be^{-ky}\right)\sin kx \sin\omega t$$

$$BC\#1 \Rightarrow u(0,y,t) = -k\left(Ae^{ky} + Be^{-ky}\right)\sin 0 \sin\omega t = 0 \qquad \Rightarrow \sin 0 = 0$$

$$BC\#2 \Rightarrow u(L,y,t) = -k\left(Ae^{ky} + Be^{-ky}\right)\sin kL \sin\omega t = 0 \qquad \Rightarrow kL = n\pi, n = 1,2,3,...$$

For the lowest mode, we select n=1, which gives $k = \pi/L$. To apply the second two boundary conditions, we find the velocity in the vertical direction:

$$v = \frac{\partial \phi}{\partial y} = k\left(Ae^{ky} - Be^{-ky}\right)\cos kx \sin\omega t$$

$$BC\#3 \Rightarrow v(x,-H,t) = k\left(Ae^{-kH} - Be^{kH}\right)\cos kx \sin\omega t = 0$$

$$\Rightarrow Ae^{-kH} = Be^{kH} = C$$

$$v = Ck\left[e^{k(y+H)} - e^{-k(y+H)}\right]\cos kx \sin\omega t$$

$$\phi = C\left[e^{k(y+H)} + e^{-k(y+H)}\right]\cos kx \sin\omega t$$

We now apply $BC\#4a$ and integrate to solve for the height of the surface disturbance, $h(x,0,t)$:

$$\frac{\partial h}{\partial t} \approx v(x,0,t) = Ck\left[e^{kH} - e^{-kH}\right]\cos kx \sin\omega t$$

$$h(x,0,t) = -\frac{ck}{\omega}\left[e^{kH} - e^{-kH}\right]\cos kx \cos\omega t$$

To apply $BC\#4b$, we use this expression for h(x,0,t) and calculate $\frac{\partial \phi}{\partial t}$:

$$\left.\frac{\partial \phi}{\partial t}\right|_{y=0} = C\omega\left[e^{kH} + e^{-kH}\right]\cos kx \cos\omega t$$

$$\left.\frac{\partial \phi}{\partial t}\right|_{y=0} + gh = const = C\omega\left[e^{kH} + e^{-kH}\right]\cos kx \cos\omega t - \frac{gCk}{\omega}\left[e^{kH} - e^{-kH}\right]\cos kx \cos\omega t$$

For this relationship to hold for all x,y, and t, the constant must be equal to zero. Therefore:

$$\omega^2 = gk\frac{e^{kH} - e^{-kH}}{e^{kH} + e^{-kH}} = gk \tanh kH$$

and hence

$$f = \frac{\omega}{2\pi} = \sqrt{\frac{g}{4\pi L} \tanh \frac{\pi H}{L}}$$